

QUANTUM RELATIONS

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ABSTRACT. We define a “quantum relation” on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ to be a weak* closed operator bimodule over its commutant \mathcal{M}' . Although this definition is framed in terms of a particular representation of \mathcal{M} , it is effectively representation independent. Quantum relations on $l^\infty(X)$ exactly correspond to subsets of X^2 , i.e., relations on X . There is also a good definition of a “measurable relation” on a measure space, to which quantum relations partially reduce in the general abelian case.

By analogy with the classical setting, we can identify structures such as quantum equivalence relations, quantum partial orders, and quantum graphs, and we can generalize Arveson’s fundamental work on weak* closed operator algebras containing a masa to these cases. We are also able to intrinsically characterize the quantum relations on \mathcal{M} in terms of families of projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$.

This paper arose out of a joint project between Greg Kuperberg and the author [10]. That project involved a new definition of quantum metrics that is suited to the von Neumann algebra setting and is especially motivated by the metric aspect of quantum error correction. In the course of that investigation, weak* closed operator bimodules over the commutant of a von Neumann algebra emerged as centrally important, and it became apparent that they were playing the role of a quantum version of relations on a set.

There is an obvious von Neumann algebra version of the notion of a relation on a set X , i.e., a subset of $X \times X$. Passing from X to a von Neumann algebra \mathcal{M} , the standard translation dictionary tells us to replace $X \times X$ with the von Neumann algebra tensor product $\mathcal{M} \overline{\otimes} \mathcal{M}$. A subset of $X \times X$ would then presumably correspond to a projection in $\mathcal{M} \overline{\otimes} \mathcal{M}$. Thus, it would be natural to take a “quantum relation” on \mathcal{M} to be a projection in $\mathcal{M} \overline{\otimes} \mathcal{M}$.

But although this definition is simple and natural, it is not particularly fruitful. If we try to identify conditions which could serve as quantum analogs of, say, reflexivity or transitivity of a relation, this line of thought becomes complicated and does not seem to lead anywhere interesting. In light of the fundamental role played in classical mathematics by the concept of a relation on a set, together with the fact that we do have robust quantum analogs of large portions of classical mathematics ([2]; see also [20]), this is rather disappointing.

A pessimistic conclusion which could be drawn is that the classical concept of a relation on a set is simply not “rigid” or “algebraic” enough to have a good quantum analog. To the contrary, we claim that there is a very good quantum analog, it is just not the obvious one.

We define a *quantum relation* on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ to be a weak* closed operator bimodule over \mathcal{M}' (Definition 2.1). Although this definition

is framed in terms of a particular representation of \mathcal{M} , it is effectively representation independent (Theorem 2.7). In the atomic abelian case, quantum relations on $l^\infty(X)$ correspond to subsets of $X \times X$, i.e., classical relations on X (Proposition 2.2). We are also able to give a reasonable definition of a “measurable relation” (Definition 1.2) to which quantum relations partially reduce in the general abelian case (Theorem 2.9 and Corollary 2.16).

Quantum analogs of such properties as reflexivity and transitivity are easily identified using the algebraic structure available in $\mathcal{B}(H)$, and this allows us to define such things as quantum partial orders and quantum graphs (Definition 2.6). These turn out to be well-known structures familiar from the standard operator algebra toolkit; for instance, a quantum preorder on \mathcal{M} is just a weak* closed operator algebra containing \mathcal{M}' .

This new point of view pays dividends. For instance, we can elegantly generalize Arveson’s fundamental results on weak* closed operator algebras containing a masa [1]. Some of this work roughly duplicates results of Erdos [5], providing a new context for that material. The advantages of our point of view are exhibited in an attractive characterization of reflexive operator space and operator system bimodules over a maximal abelian von Neumann algebra (Theorem 2.22).

While much of the content of Sections 2.4 and 2.5 could actually be derived from Arveson’s work, in some ways the new point of view provides a broader perspective that could be useful. For example, we can define a pullback of quantum relations (Proposition 2.25); this is not possible in the setting of weak* closed operator algebras because pullbacks are not compatible with products in general. (Thus, the pullback of a quantum preorder need not be a quantum preorder.)

Our most substantial result is an intrinsic characterization of quantum relations. Given a von Neumann algebra \mathcal{M} , let \mathcal{P} be the set of projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$, equipped with the restriction of the weak operator topology. We define an *intrinsic quantum relation* on \mathcal{M} to be an open subset $\mathcal{R} \subset \mathcal{P} \times \mathcal{P}$ satisfying

- (i) $(0, 0) \notin \mathcal{R}$
- (ii) $(\bigvee P_\lambda, \bigvee Q_\kappa) \in \mathcal{R} \Leftrightarrow \text{some } (P_\lambda, Q_\kappa) \in \mathcal{R}$
- (iii) $(P, [BQ]) \in \mathcal{R} \Leftrightarrow ([B^*P], Q) \in \mathcal{R}$

for all projections $P, Q, P_\lambda, Q_\kappa \in \mathcal{P}$ and all $B \in I \otimes \mathcal{B}(l^2)$. Here square brackets denote range projection. We prove that intrinsic quantum relations on \mathcal{M} naturally correspond to quantum relations on \mathcal{M} (Theorem 2.32).

To illustrate the tractability of quantum relations, we introduce a notion of translation invariance for quantum relations on quantum torus von Neumann algebras (Definition 2.39) and characterize the structure of quantum relations on quantum tori with this property (Theorem 2.41 and Corollary 2.42).

We work with complex scalars throughout.

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1. MEASURABLE RELATIONS

A relation on a set X is a subset of X^2 . Before discussing a noncommutative version of this notion, we first consider the measurable setting since the idea of a “measurable relation” is already interesting. Some readers may prefer to skip to Section 2 and refer back to this section as needed.

1.1. Basic definitions. The obvious definition of a measurable relation on a measure space (X, μ) would be a measurable subset of X^2 up to modification on a null set. But if (X, μ) is nonatomic then under this definition the condition that a relation be reflexive becomes vacuous, because the diagonal of $X \times X$ has measure zero. The notion of transitivity also becomes problematic. Therefore we seek a better-behaved candidate to play the role of a measurable relation. Our proposed definition is more complicated than the one suggested above, but it has elegant properties and generates a robust theory.

To avoid pathology we assume that all measure spaces are *finitely decomposable*. This means that the space X can be partitioned into a (possibly uncountable) family of finite measure subspaces X_λ such that a set $S \subseteq X$ is measurable if and only if its intersection with each X_λ is measurable, in which case $\mu(S) = \sum \mu(S \cap X_\lambda)$ ([18], Definition 6.1.1). Finite decomposability is a generalization of σ -finiteness. Counting measure on any set is also finitely decomposable. The spaces $L^\infty(X, \mu)$ with (X, μ) finitely decomposable are precisely the abelian von Neumann algebras.

If (X, μ) is finitely decomposable then the projections in the abelian von Neumann algebra $L^\infty(X, \mu)$ constitute a complete lattice. These projections are precisely the characteristic functions χ_S of measurable subsets $S \subseteq X$ up to null sets. Thus, working with projections in $L^\infty(X, \mu)$ is a convenient way to factor out equivalence modulo null sets.

Our approach will be to model a measurable relation by specifying which pairs of projections belong to it. This differs from the classical pointwise notion but the two can be made equivalent in the atomic case if we adopt an appropriate axiom specifying how the pairs of projections belonging to the relation must cohere. The suitable coherence axiom $(*)$ in Definition 1.2 below) is motivated by the following probably well-known result.

Proposition 1.1. *Let (X, μ) be a finitely decomposable measure space and let \mathcal{F} be a family of nonzero projections in $L^\infty(X, \mu)$ with the property that for any family of nonzero projections $\{p_\lambda\}$ in $L^\infty(X, \mu)$ we have*

$$\bigvee p_\lambda \in \mathcal{F} \quad \Leftrightarrow \quad \text{some } p_\lambda \in \mathcal{F}.$$

Then there is a projection $p \in L^\infty(X, \mu)$ such that $q \in \mathcal{F} \Leftrightarrow pq \neq 0$.

Proof. Let r be the join of all the projections that are not in \mathcal{F} . Then the displayed condition implies that r is not in \mathcal{F} . Moreover, if q is any projection less than r then $r \vee q = r \notin \mathcal{F}$ implies $q \notin \mathcal{F}$. So a projection is not in \mathcal{F} if and only if it lies below r . Letting $p = 1 - r$, we then have $q \in \mathcal{F} \Leftrightarrow pq \neq 0$. \square

Since the converse of Proposition 1.1 is trivial — for any projection p the set of projections q such that $pq \neq 0$ does have the stated property — this result gives us a (somewhat roundabout) characterization of the measurable subsets of X up to null sets. Namely, having a measurable subset is the same as having a family \mathcal{F} of nonzero projections with the property that $\bigvee p_\lambda \in \mathcal{F} \Leftrightarrow \text{some } p_\lambda \in \mathcal{F}$. (Such a family is just a proper complete filter of the lattice of projections in $L^\infty(X, \mu)$.) This makes the following definition plausible.

Definition 1.2. Let (X, μ) be a finitely decomposable measure space. A *measurable relation* on X is a family \mathcal{R} of ordered pairs of nonzero projections in $L^\infty(X, \mu)$ such that

$$\left(\bigvee p_\lambda, \bigvee q_\kappa \right) \in \mathcal{R} \quad \Leftrightarrow \quad \text{some } (p_\lambda, q_\kappa) \in \mathcal{R} \quad (*)$$

for any pair of families of nonzero projections $\{p_\lambda\}$ and $\{q_\kappa\}$. Equivalently, we can impose the two conditions

$$p' \leq p, \quad q' \leq q, \quad (p', q') \in \mathcal{R} \quad \Rightarrow \quad (p, q) \in \mathcal{R} \quad (*')$$

and

$$\left(\bigvee p_\lambda, \bigvee q_\kappa \right) \in \mathcal{R} \quad \Rightarrow \quad \text{some } (p_\lambda, q_\kappa) \in \mathcal{R}. \quad (**')$$

(It is easy to check that $(*)'$ is equivalent to the reverse implication in $(*)$, using the fact that $p' \leq p \Leftrightarrow p' \vee p = p$.)

The generalization to a measurable relation on a pair of finitely decomposable measure spaces (X, μ) and (Y, ν) would be a family of ordered pairs of nonzero projections $p \in L^\infty(X, \mu)$ and $q \in L^\infty(Y, \nu)$ satisfying condition $(*)$ (or, alternatively, conditions $(*)'$ and $(**')$) in Definition 1.2. We need not develop this more general notion separately since measurable relations on X and Y can be identified with measurable relations on the disjoint union $X \amalg Y$ that do not contain the pairs (χ_X, χ_X) , (χ_Y, χ_X) , or (χ_Y, χ_Y) (i.e., that live in the $X \times Y$ corner of $(X \amalg Y)^2$).

The intuition behind Definition 1.2 is that a pair of projections (χ_S, χ_T) belongs to \mathcal{R} if and only if some point in S is related to some point in T . This pointwise condition is not meaningful in the general measurable case, but we will now show that in the atomic case it effectively reduces measurable relations on X to subsets of X^2 . Let $e_x = \chi_{\{x\}}$ be the characteristic function of singleton x .

Proposition 1.3. Let μ be counting measure on a set X . If R is a relation on X then

$$\begin{aligned} \mathcal{R}_R &= \{(\chi_S, \chi_T) : (x, y) \in R \text{ for some } x \in S, y \in T\} \\ &= \{(\chi_S, \chi_T) : (S \times T) \cap R \neq \emptyset\} \end{aligned}$$

is a measurable relation on X ; conversely, if \mathcal{R} is a measurable relation on X then

$$R_{\mathcal{R}} = \{(x, y) \in X^2 : (e_x, e_y) \in \mathcal{R}\}$$

is a relation on X . The two constructions are inverse to each other.

Proof. \mathcal{R}_R is a measurable relation because $(\bigcup S_\lambda) \times (\bigcup T_\kappa) = \bigcup_{\lambda, \kappa} (S_\lambda \times T_\kappa)$ intersects $R \subseteq X^2$ if and only if $S_\lambda \times T_\kappa$ intersects R for some λ, κ . The fact that \mathcal{R}_R is a relation is trivial. Now let R be a relation, let $\mathcal{R} = \mathcal{R}_R$, and let $\tilde{R} = R_{\mathcal{R}}$. Then

$$(x, y) \in R \Leftrightarrow (e_x, e_y) \in \mathcal{R} \Leftrightarrow (x, y) \in \tilde{R}$$

for all $x, y \in X$, so $R = \tilde{R}$. Finally, let \mathcal{R} be a measurable relation, let $R = R_{\mathcal{R}}$, and let $\tilde{\mathcal{R}} = \mathcal{R}_R$. Since \mathcal{R} is a measurable relation and $\chi_S = \bigvee_{x \in S} e_x$, $\chi_T = \bigvee_{y \in T} e_y$ we have

$$\begin{aligned} (\chi_S, \chi_T) \in \mathcal{R} &\Leftrightarrow (e_x, e_y) \in \mathcal{R} \text{ for some } x \in S, y \in T \\ &\Leftrightarrow (x, y) \in R \text{ for some } x \in S, y \in T \\ &\Leftrightarrow (\chi_S, \chi_T) \in \tilde{\mathcal{R}} \end{aligned}$$

for all nonzero projections χ_S and χ_T . So $\mathcal{R} = \tilde{\mathcal{R}}$. \square

We also want to mention the following alternative characterization of measurable relations. This will be used in Section 2.5 to establish a connection with operator space reflexivity.

Proposition 1.4. *Let (X, μ) be a finitely decomposable measure space. If \mathcal{R} is a measurable relation on X then the map*

$$\phi_{\mathcal{R}} : q \mapsto 1 - \bigvee \{p : (p, q) \notin \mathcal{R}\},$$

from the set of projections in $L^\infty(X, \mu)$ to itself, takes 0 to 0 and preserves arbitrary joins. If ϕ is a map from the set of projections in $L^\infty(X, \mu)$ to itself that takes 0 to 0 and preserves arbitrary joins then

$$\mathcal{R}_\phi = \{(p, q) : p\phi(q) \neq 0\}$$

is a measurable relation on X . The two constructions are inverse to each other.

Proof. We start with a simple observation. Let \mathcal{R} be a measurable relation and let p and q be projections in $L^\infty(X, \mu)$. It is immediate from the definition of measurable relations that $(1 - \phi_{\mathcal{R}}(q), q) \notin \mathcal{R}$, so if $p\phi_{\mathcal{R}}(q) = 0$ then $p \leq 1 - \phi_{\mathcal{R}}(q)$ and hence $(p, q) \notin \mathcal{R}$. Conversely, if $p\phi_{\mathcal{R}}(q) \neq 0$ then $(p, q) \in \mathcal{R}$ by the definition of $\phi_{\mathcal{R}}$. So $(p, q) \in \mathcal{R}$ if and only if $p\phi_{\mathcal{R}}(q) \neq 0$.

It is clear that $\phi_{\mathcal{R}}(0) = 0$. Let $\{q_\kappa\}$ be any family of projections in $L^\infty(X, \mu)$. If $p \cdot \bigvee \phi_{\mathcal{R}}(q_\kappa) = 0$ then $p\phi_{\mathcal{R}}(q_\kappa) = 0$ for all κ , so that $(p, q_\kappa) \notin \mathcal{R}$ for all κ , which implies that $(p, \bigvee q_\kappa) \notin \mathcal{R}$. Conversely, if $p \cdot \bigvee \phi_{\mathcal{R}}(q_\kappa) \neq 0$ then $p\phi_{\mathcal{R}}(q_\kappa) \neq 0$ for some κ , so that $(p, q_\kappa) \in \mathcal{R}$ for that κ , which implies that $(p, \bigvee q_\kappa) \in \mathcal{R}$. So we have shown that $(p, \bigvee q_\kappa) \in \mathcal{R}$ if and only if $p \cdot \bigvee \phi_{\mathcal{R}}(q_\kappa) \neq 0$. By the preceding paragraph we also have that $(p, \bigvee q_\kappa) \in \mathcal{R}$ if and only if $p\phi_{\mathcal{R}}(\bigvee q_\kappa) \neq 0$, and this implies that $\phi_{\mathcal{R}}(\bigvee q_\kappa) = \bigvee \phi_{\mathcal{R}}(q_\kappa)$. Thus $\phi_{\mathcal{R}}$ takes 0 to 0 and preserves arbitrary joins.

Now let ϕ be any map which takes 0 to 0 and preserves arbitrary joins and let $\{p_\lambda\}$ and $\{q_\kappa\}$ be families of nonzero projections. Then

$$\begin{aligned} (\bigvee p_\lambda, \bigvee q_\kappa) \in \mathcal{R}_\phi &\Leftrightarrow (\bigvee p_\lambda) \phi(\bigvee q_\kappa) \neq 0 \\ &\Leftrightarrow (\bigvee p_\lambda) (\bigvee \phi(q_\kappa)) \neq 0 \\ &\Leftrightarrow p_\lambda \phi(q_\kappa) \neq 0 \text{ for some } \lambda, \kappa \\ &\Leftrightarrow (p_\lambda, q_\kappa) \in \mathcal{R}_\phi \text{ for some } \lambda, \kappa. \end{aligned}$$

Also, since $\phi(0) = 0$ it is clear that $(p, q) \notin \mathcal{R}_\phi$ if either p or q is 0. So \mathcal{R}_ϕ is a measurable relation.

The fact that $\mathcal{R} = \mathcal{R}_{\phi_{\mathcal{R}}}$ follows immediately from the observation made in the first paragraph of the proof. The identity $\phi = \phi_{\mathcal{R}_\phi}$ follows from the fact that $\bigvee \{p : p\phi(q) = 0\} = 1 - \phi(q)$. \square

1.2. Constructions with measurable relations. The following are basic constructions with measurable relations.

Proposition 1.5. *Let (X, μ) be a finitely decomposable measure space.*

(a) *The set of pairs of projections p and q in $L^\infty(X, \mu)$ such that $pq \neq 0$ is a measurable relation on X .*

(b) *If \mathcal{R} is a measurable relation on X then so is $\{(q, p) : (p, q) \in \mathcal{R}\}$.*

(c) *If \mathcal{R} and \mathcal{R}' are measurable relations on X then a pair of nonzero projections (p, r) satisfies*

for every projection q , either $(p, q) \in \mathcal{R}$ or $(1 - q, r) \in \mathcal{R}'$

if and only if it satisfies

*there exists a nonzero projection q such that $(p, q') \in \mathcal{R}$ and $(q', r) \in \mathcal{R}'$
for every nonzero $q' \leq q$*

and the set of all pairs satisfying these conditions constitutes a measurable relation.

(d) *Any union of measurable relations on X is a measurable relation on X .*

(e) *If \mathcal{R} is a measurable relation on a finitely decomposable measure space (Y, ν) and $\phi : L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ is a unital weak* continuous *-homomorphism then*

$$\phi^*(\mathcal{R}) = \{(p, q) : (\phi(p), \phi(q)) \in \mathcal{R}\}$$

is a measurable relation on X .

Proof. Parts (a), (b), (d), and (e) are easy. For part (c), fix measurable relations \mathcal{R} and \mathcal{R}' and nonzero projections p and r . Suppose first that every projection q satisfies either $(p, q) \in \mathcal{R}$ or $(1 - q, r) \in \mathcal{R}'$. Let r' be the join of all the projections r'' such that $(p, r'') \notin \mathcal{R}$ and let p' be the join of all the projections p'' such that $(p'', r) \notin \mathcal{R}'$. If $p' \vee r' = 1$ then $q = r'$ falsifies our assumption on the pair (p, r) . Therefore $q = 1 - p' \vee r'$ is a nonzero projection such that $(p, q') \in \mathcal{R}$ and $(q', r) \in \mathcal{R}'$ for every nonzero $q' \leq q$.

Next, suppose there exists a projection q satisfying $(p, q) \notin \mathcal{R}$ and $(1 - q, r) \notin \mathcal{R}'$. Then every nonzero projection q' either satisfies $q'q \neq 0$, in which case q' lies over a nonzero projection $q'q$ with $(p, q'q) \notin \mathcal{R}$, or else $q'(1 - q) \neq 0$, in which case q' lies over a nonzero projection $q'(1 - q)$ with $(q'(1 - q), r) \notin \mathcal{R}'$. This completes the proof that the two conditions are equivalent.

For the second assertion in part (c), we verify conditions $(*)$ and $(**)$ in Definition 1.2. The first condition is trivial. For the second, suppose that for every λ and κ there exists a projection $q_{\lambda, \kappa}$ such that $(p_\lambda, q_{\lambda, \kappa}) \notin \mathcal{R}$ and $(1 - q_{\lambda, \kappa}, r_\kappa) \notin \mathcal{R}'$. The projection

$$q = \bigwedge_{\lambda} \bigvee_{\kappa} q_{\lambda, \kappa}$$

then satisfies $(\bigvee p_\lambda, q) \notin \mathcal{R}$ and $(1 - q, \bigvee r_\kappa) \notin \mathcal{R}'$ (because $1 - q = \bigvee_{\lambda} \bigwedge_{\kappa} (1 - q_{\lambda, \kappa})$). This shows that condition $(**)$ from Definition 1.2 holds. This completes the proof. \square

This justifies the following definition.

Definition 1.6. Let (X, μ) be a finitely decomposable measure space.

- (a) The *diagonal measurable relation* Δ on X is defined by setting $(p, q) \in \Delta$ if $pq \neq 0$.
- (b) The *transpose* of a measurable relation \mathcal{R} is the measurable relation $\mathcal{R}^T = \{(q, p) : (p, q) \in \mathcal{R}\}$.
- (c) The *product* of two measurable relations \mathcal{R} and \mathcal{R}' is the measurable relation

$$\mathcal{R} \cdot \mathcal{R}' = \{(p, r) : \text{either condition in Proposition 1.5 (c) holds}\}.$$

- (d) A measurable relation \mathcal{R} on X is

- (i) *reflexive* if $\Delta \subseteq \mathcal{R}$
- (ii) *symmetric* if $\mathcal{R}^T = \mathcal{R}$
- (iii) *antisymmetric* if $\mathcal{R} \wedge \mathcal{R}^T \subseteq \Delta$
- (iv) *transitive* if $\mathcal{R}^2 \subseteq \mathcal{R}$.

In (d) (iii), $\mathcal{R} \wedge \mathcal{R}^T$ denotes the greatest lower bound of \mathcal{R} and \mathcal{R}^T under the partial ordering \subseteq . The set of measurable relations is a complete lattice by Proposition 1.5 (d).

The diagonal relation intuitively consists of those pairs (χ_S, χ_T) such that $S \times T$ intersects the diagonal in a set of positive measure. Of course, this intuition is not accurate since the diagonal could have measure zero. Notice that under our definition the diagonal relation on any nonzero measure space X is different from the zero relation, even if the set-theoretic diagonal of X^2 has measure zero.

We note that pullbacks (Proposition 1.5 (e)) are not compatible with products; indeed, the atomic version of this statement (with reversed arrows) already fails. For example, let $X = \{x, y, z\}$ and $Y = \{x, y_1, y_2, z\}$ and define $f : Y \rightarrow X$ by $f(x) = x$, $f(y_i) = y$ ($i = 1, 2$), and $f(z) = z$. Then the relations $R = \{(x, y_1)\}$ and $R' = \{(y_2, z)\}$ on Y satisfy $R \cdot R' = \emptyset$, so that $f_*(R \cdot R') = \emptyset$, but the product $f_*(R) \cdot f_*(R')$ of their pushforwards is $\{(x, z)\}$.

We also note that the above definitions generalize the corresponding classical constructions with relations on sets.

Proposition 1.7. Let μ be counting measure on a set X , let R_1, R_2 , and R_3 be relations on X , and let $\mathcal{R}_i = \mathcal{R}_{R_i}$ ($i = 1, 2, 3$) be the corresponding measurable relations as in Proposition 1.3. Then

- (a) $R_1 \subseteq R_2 \Leftrightarrow \mathcal{R}_1 \subseteq \mathcal{R}_2$;
- (b) R_1 is the diagonal relation $\Leftrightarrow \mathcal{R}_1$ is the diagonal measurable relation;
- (c) R_1 is the transpose of $R_2 \Leftrightarrow \mathcal{R}_1$ is the transpose of \mathcal{R}_2 ; and
- (d) R_3 is the product of R_1 and $R_2 \Leftrightarrow \mathcal{R}_3$ is the product of \mathcal{R}_1 and \mathcal{R}_2 .

The proof of this proposition is straightforward.

Using Definition 1.6 we can define measurable versions of equivalence relations, preorders, partial orders, and graphs:

Definition 1.8. Let (X, μ) be a finitely decomposable measure space.

- (a) A *measurable equivalence relation* on X is a reflexive, symmetric, transitive measurable relation on X .
- (b) A *measurable preorder* on X is a reflexive, transitive measurable relation on X .
- (c) A *measurable partial order* on X is a reflexive, antisymmetric, transitive measurable relation on X .
- (d) A *measurable graph* on X is a reflexive, symmetric measurable relation on X .

It follows from Proposition 1.7 that the preceding definitions generalize the corresponding classical definitions.

Part (d) requires some explanation. A (simple) graph is usually defined to be a vertex set V together with a family of 2-element subsets of V . But the same information determines and is determined by a reflexive, symmetric relation on V and so a graph may equivalently be defined as a set equipped with such a relation. This justifies our definition of a measurable graph.

The expected definition of a measurable equivalence relation is probably a von Neumann subalgebra of $L^\infty(X, \mu)$, or equivalently ([14], Theorem II.4.24) a complete Boolean subalgebra of the Boolean algebra of projections in $L^\infty(X, \mu)$. We will now show that this is equivalent to our definition, and more generally, that measurable preorders correspond to complete 0,1-sublattices of the lattice of projections in $L^\infty(X, \mu)$.

Given a measurable relation \mathcal{R} on X , say that $S \subseteq X$ is a *lower set* if $(1 - \chi_S, \chi_S) \notin \mathcal{R}$. In the atomic case this means that all points below any point in S belong to S .

Theorem 1.9. *Let (X, μ) be a finitely decomposable measure space. If \mathcal{R} is a measurable preorder on X then*

$$\begin{aligned} \mathcal{L}_{\mathcal{R}} &= \{p \in L^\infty(X, \mu) : p \text{ is a projection and } (1 - p, p) \notin \mathcal{R}\} \\ &= \{\chi_S : S \subseteq X \text{ is a lower set}\} \end{aligned}$$

is a complete 0,1-sublattice of the lattice of projections in $L^\infty(X, \mu)$. If \mathcal{L} is a complete 0,1-sublattice of the lattice of projections in $L^\infty(X, \mu)$ then

$$\mathcal{R}_{\mathcal{L}} = \{(p, q) : pq' \neq 0 \text{ for all } q' \in \mathcal{L} \text{ with } q' \geq q\}$$

is a measurable preorder on X . The two constructions are inverse to each other. This correspondence between measurable preorders and complete 0,1-sublattices restricts to a correspondence between measurable equivalence relations and complete Boolean subalgebras.

Proof. Let \mathcal{R} be a measurable preorder on X . It is clear that 0 and 1 belong to $\mathcal{L}_{\mathcal{R}}$, and if \mathcal{R} is symmetric then $\mathcal{L}_{\mathcal{R}}$ is closed under complementation. Now let $\{p_\lambda\}$ be any family of projections in $\mathcal{L}_{\mathcal{R}}$ and let $p = \bigvee p_\lambda$. Then $(1 - p_\lambda, p_\lambda) \notin \mathcal{R}$ for every λ , hence $(1 - p, p_\lambda) \notin \mathcal{R}$ for every λ (since $1 - p \leq 1 - p_\lambda$), hence $(1 - p, p) \notin \mathcal{R}$ (since $p = \bigvee p_\lambda$). This shows that $p \in \mathcal{L}_{\mathcal{R}}$. Letting $q = \bigwedge p_\lambda$, we also have that $(1 - p_\lambda, p_\lambda) \notin \mathcal{R}$ for every λ implies $(1 - p_\lambda, q) \notin \mathcal{R}$ for every λ . Since $1 - q = \bigvee (1 - p_\lambda)$, it follows that $(1 - q, q) \notin \mathcal{R}$, i.e., $q \in \mathcal{L}_{\mathcal{R}}$. So $\mathcal{L}_{\mathcal{R}}$ is a complete 0,1-sublattice of the lattice of projections in $L^\infty(X, \mu)$, and it is a Boolean algebra if \mathcal{R} is a measurable equivalence relation.

Next let \mathcal{L} be any complete 0,1-sublattice of the lattice of projections. We first check that $\mathcal{R}_{\mathcal{L}}$ satisfies the pair of conditions stated in Definition 1.2. Condition $(*)'$ is trivial. For condition $(*)''$, let $\{p_\lambda\}$ and $\{q_\kappa\}$ be any families of nonzero projections and suppose $(p_\lambda, q_\kappa) \notin \mathcal{R}_{\mathcal{L}}$ for all λ and κ . There must exist $q_{\lambda, \kappa} \in \mathcal{L}$ with $q_\kappa \leq q_{\lambda, \kappa}$ and $p_\lambda q_{\lambda, \kappa} = 0$. Then

$$q' = \bigvee_{\kappa} \bigwedge_{\lambda} q_{\lambda, \kappa} \in \mathcal{L}$$

satisfies $\bigvee q_\kappa \leq q'$ and $(\bigvee p_\lambda)q' = 0$, which shows that $(\bigvee p_\lambda, \bigvee q_\kappa) \notin \mathcal{R}_{\mathcal{L}}$. This verifies condition $(*)''$, so $\mathcal{R}_{\mathcal{L}}$ is a measurable relation.

Reflexivity of $\mathcal{R}_{\mathcal{L}}$ is easy, as is symmetry if \mathcal{L} is Boolean. To verify transitivity, let p and r be nonzero projections such that for every projection q , either $(p, q) \in \mathcal{R}_{\mathcal{L}}$ or $(1 - q, r) \in \mathcal{R}_{\mathcal{L}}$; we must show that $(p, r) \in \mathcal{R}_{\mathcal{L}}$. Let r' be a projection in \mathcal{L} such that $r \leq r'$. Then $(1 - r', r) \notin \mathcal{R}_{\mathcal{L}}$ directly from the definition of $\mathcal{R}_{\mathcal{L}}$ since r' is a projection in \mathcal{L} that contains r and is disjoint from $1 - r'$. So the condition we assumed on p and r yields $(p, r') \in \mathcal{R}_{\mathcal{L}}$. From this it follows that $pr' \neq 0$ (since r' is a projection in \mathcal{L} with $r' \leq r'$), and we conclude that $(p, r) \in \mathcal{R}_{\mathcal{L}}$, as desired. Thus, we have shown that $\mathcal{R}_{\mathcal{L}}$ is a measurable preorder, and that it is a measurable equivalence relation if \mathcal{L} is Boolean.

Now let \mathcal{R} be a measurable preorder, let $\mathcal{L} = \mathcal{L}_{\mathcal{R}}$, and let $\tilde{\mathcal{R}} = \mathcal{R}_{\mathcal{L}}$. If $(p, q) \in \mathcal{R}$ then for any $q' \in \mathcal{L}$ with $q \leq q'$ we must have $(p, q') \in \mathcal{R}$, which by the definition of \mathcal{L} implies $pq' \neq 0$ (since $(1 - q', q') \notin \mathcal{R}$). Thus $(p, q) \in \mathcal{R}$ implies $(p, q) \in \tilde{\mathcal{R}}$. For the reverse inclusion, suppose $(p, q) \notin \mathcal{R}$, let p' be the join of all the projections r such that $(r, q) \notin \mathcal{R}$, and let $q' = 1 - p'$; we claim that $q \leq q'$ and $q' \in \mathcal{L}$. Since $p \leq p'$ and hence $pq' = 0$, this will verify that $(p, q) \notin \tilde{\mathcal{R}}$. First, if $(r, q) \notin \mathcal{R}$ then $rq = 0$ by reflexivity of \mathcal{R} , and this shows that $p'q = 0$, i.e., $q \leq q'$. To see that $q' \in \mathcal{L}$ we must show that $(p', q') \notin \mathcal{R}$. Let r' be the join of all the projections r such that $(p', r) \notin \mathcal{R}$. We must have $r' \leq q'$ by reflexivity. But if $r' \neq q'$ then every nonzero projection $r \leq q' - r'$ satisfies $(p', r) \in \mathcal{R}$ (since $rr' = 0$) and $(r, q) \in \mathcal{R}$ (since $r \leq q' = 1 - p'$); by transitivity, this implies that $(p', q) \in \mathcal{R}$, a contradiction. Therefore $r' = q'$ and hence $(p', q') \notin \mathcal{R}$. This completes the proof of the reverse inclusion.

Finally, let \mathcal{L} be a complete 0,1-lattice of projections, let $\mathcal{R} = \mathcal{R}_{\mathcal{L}}$, and let $\tilde{\mathcal{L}} = \mathcal{L}_{\mathcal{R}}$. Then it is trivial that a projection p belongs to \mathcal{L} if and only if there exists $p' \in \mathcal{L}$ such that $p \leq p'$ and $(1 - p)p' = 0$. Thus $p \in \tilde{\mathcal{L}} \Leftrightarrow (1 - p, p) \notin \mathcal{R} \Leftrightarrow p \in \mathcal{L}$. So $\mathcal{L} = \tilde{\mathcal{L}}$. \square

1.3. Conversion to classical relations. In the general measurable setting the correspondence identified in Proposition 1.3 remains partially valid: if R is a measurable subset of X^2 then

$$\mathcal{R} = \{(\chi_S, \chi_T) : (S \times T) \cap R \text{ is nonnull}\}$$

is a measurable relation on X . However, this is of limited value because, first, there typically exist measurable relations that do not derive from subsets of X^2 in the above manner; the diagonal measurable relation (Definition 1.6 (a)) on any nonatomic measure space is an example of this phenomenon. Second, distinct subsets of X^2 (that is, distinct even modulo null sets) can give rise to the same measurable relation. For example, there exists a measurable subset R of $[0, 1]^2$, of measure strictly less than 1, that has positive measure intersection with every positive measure subset of the form $S \times T$. The measurable relation derived from such a set in the manner suggested above is the same as the measurable relation derived from the full relation $[0, 1]^2$. Such a set R can be constructed as the set of pairs $(x, y) \in [0, 1]^2$ such that $x - y$ belongs to a dense open subset U of $[-1, 1]$ with measure strictly less than 2. For if S and T are any positive measure subsets of $[0, 1]$, let s be a Lebesgue point of S and t a Lebesgue point of T , and find $\epsilon > 0$ such that

$$\mu((s - \epsilon, s + \epsilon) \cap S), \mu((t - \epsilon, t + \epsilon) \cap T) > \frac{3\epsilon}{2}.$$

Then for all $a \in [-\epsilon, \epsilon]$ we have

$$\mu((S + t + a) \cap I) > \frac{\epsilon}{2} \quad \text{and} \quad \mu((T + s) \cap I) > \frac{3\epsilon}{2}$$

where I is the interval $(s + t - \epsilon, s + t + \epsilon)$ of length 2ϵ . It follows that $\mu((S + t + a) \cap (T + s)) > 0$ for all $a \in [-\epsilon, \epsilon]$. Since U is dense in $[-1, 1]$, find $a \in [-\epsilon, \epsilon]$ such that $s - t - a \in U$, and let u be a Lebesgue point of $(S + t + a) \cap (T + s)$; then $s' = u - t - a$ and $t' = u - s$ are respectively Lebesgue points of S and T , and $s' - t' = s - t - a \in U$. It follows that for sufficiently small δ the sets $(s' - \delta, s' + \delta) \cap S$ and $(t' - \delta, t' + \delta) \cap T$ have positive measure and their product is contained in R . Thus $(S \times T) \cap R$ has positive measure, as claimed.

We can, however, always convert measurable relations to pointwise relations in the following standard way. Let (X, μ) be a finitely decomposable measure space and recall that the *carrier space* Ω of $L^\infty(X, \mu)$ is the set of nonzero homomorphisms from $L^\infty(X, \mu)$ into \mathbf{C} , that Ω is a compact Hausdorff space with weak* topology inherited from the dual of $L^\infty(X, \mu)$, and that there is a natural isomorphism $\Phi : L^\infty(X, \mu) \cong C(\Omega)$ ([14], Theorem I.4.4).

(If μ is σ -finite then we can go further: letting $g \in L^1(X, \mu)$ be positive and nowhere-zero, integration against g on $L^\infty(X, \mu)$ corresponds to a bounded positive linear functional on $C(\Omega)$, and hence is given by integration against a regular Borel measure ν on Ω . That is,

$$\int fg \, d\mu = \int \Phi(f) \, d\nu$$

for all $f \in L^\infty(X, \mu)$. Then $C(\Omega) \cong L^\infty(\Omega, \nu)$ by essentially the identity map, so that we can regard Φ as an isomorphism between $L^\infty(X, \mu)$ and $L^\infty(\Omega, \nu)$ ([14], Theorem III.1.18).)

If p is a projection in $L^\infty(X, \mu)$ and $\phi \in \Omega$ then $\phi(p) = 0$ or 1 . Now if \mathcal{R} is a measurable relation on X then we can define a corresponding relation $R_{\mathcal{R}}$ on Ω by setting $(\phi, \psi) \in R_{\mathcal{R}}$ if $(p, q) \in \mathcal{R}$ for every pair of projections p and q in $L^\infty(X, \mu)$ such that $\phi(p) = \psi(q) = 1$.

Theorem 1.10. *Let (X, μ) be a σ -finite measure space, let $\mathcal{R}_1, \mathcal{R}_2$, and \mathcal{R}_3 be measurable relations on X , and let $R_i = R_{\mathcal{R}_i}$ ($i = 1, 2, 3$) be the corresponding relations on Ω as defined above. Then*

- (a) R_1 is a closed relation on Ω
- (b) $\mathcal{R}_1 \subseteq \mathcal{R}_2 \Leftrightarrow R_1 \subseteq R_2$
- (c) \mathcal{R}_1 is the diagonal measurable relation $\Leftrightarrow R_1$ is the diagonal relation
- (d) \mathcal{R}_1 is the transpose of $\mathcal{R}_2 \Leftrightarrow R_1$ is the transpose of R_2
- (e) \mathcal{R}_3 is the product of \mathcal{R}_1 and $\mathcal{R}_2 \Leftrightarrow R_3$ is the product of R_1 and R_2 .

Proof. R_1 is closed because if $\phi_\lambda \rightarrow \phi$ and $\psi_\lambda \rightarrow \psi$ weak* with $(\phi_\lambda, \psi_\lambda) \in R_1$ for all λ , and p and q are projections with $\phi(p) = \psi(q) = 1$, then eventually $\phi_\lambda(p) = \psi_\lambda(q) = 1$, which implies that $(p, q) \in \mathcal{R}_1$; this shows that $(\phi, \psi) \in R_1$.

It is clear that $\mathcal{R}_1 \subseteq \mathcal{R}_2$ implies $R_1 \subseteq R_2$. For the converse, we claim that $(p, q) \in \mathcal{R}_1$ if and only if there exist $\phi, \psi \in \Omega$ such that $\phi(p) = \psi(q) = 1$ and $(\phi, \psi) \in R_1$; this obviously yields that $R_1 \subseteq R_2$ implies $\mathcal{R}_1 \subseteq \mathcal{R}_2$. The reverse implication in the claim is immediate from the definition of R_1 . For the forward implication, suppose $(p, q) \in \mathcal{R}_1$, say $p = \chi_S$ and $q = \chi_T$. For any finite partitions $S = \{S_1, \dots, S_m\}$ of S and $T = \{T_1, \dots, T_n\}$ of T , choose $1 \leq i \leq m$ and $1 \leq j \leq n$

such that $(\chi_{S_i}, \chi_{T_j}) \in \mathcal{R}_1$ and define $\phi_{\mathcal{S}, \mathcal{T}}, \psi_{\mathcal{S}, \mathcal{T}} : L^\infty(X, \mu) \rightarrow \mathbf{C}$ by

$$\phi_{\mathcal{S}, \mathcal{T}}(f) = \frac{1}{\mu(S_i)} \int_{S_i} f d\mu \quad \text{and} \quad \psi_{\mathcal{S}, \mathcal{T}}(f) = \frac{1}{\mu(T_j)} \int_{T_j} f d\mu.$$

The limit (ϕ, ψ) of any weak* convergent subnet of the net $\{(\phi_{\mathcal{S}, \mathcal{T}}, \psi_{\mathcal{S}, \mathcal{T}})\}$ will then be a pair of complex homomorphisms with the property that $\phi(p) = \psi(q) = 1$. To see that $(\phi, \psi) \in R_1$ let p' and q' be any projections in $L^\infty(X, \mu)$ such that $\phi(p') = \psi(q') = 1$. Say $p' = \chi_{S'}$ and $q' = \chi_{T'}$. Then the net $\{(\phi_{\mathcal{S}, \mathcal{T}}, \psi_{\mathcal{S}, \mathcal{T}})\}$ has the property that eventually $S' \cap S$ is a finite union of blocks in the partition \mathcal{S} , and similarly for $T' \cap T$, so that $\phi(p') = \psi(q') = 1$ implies that in the subnet that converges to (ϕ, ψ) , eventually $S_i \subseteq S'$ and $T_j \subseteq T'$ for the choice of i and j used to define $\phi_{\mathcal{S}, \mathcal{T}}$ and $\psi_{\mathcal{S}, \mathcal{T}}$. This shows that $(p', q') \in \mathcal{R}_1$, and we conclude that $(\phi, \psi) \in R_1$. This completes the proof that $R_1 \subseteq R_2$ implies $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

Observe that part (b) implies $\mathcal{R}_1 = \mathcal{R}_2 \Leftrightarrow R_1 = R_2$.

Next, suppose \mathcal{R}_1 is the diagonal measurable relation on X and let $\phi \in \Omega$. For any projections p and q with $\phi(p) = \phi(q) = 1$ we must have $\phi(pq) = 1$ since ϕ is a homomorphism, and therefore $pq \neq 0$. Thus $(p, q) \in \mathcal{R}_1$ for any p and q with $\phi(p) = \phi(q) = 1$, so that $(\phi, \phi) \in R_1$. Conversely, if $\phi, \psi \in \Omega$ are distinct then there is a projection p such that $\phi(p) \neq \psi(p)$. Say $\phi(p) = 1$ and $\psi(p) = 0$. Then $\psi(1-p) = 1$, and the pair $(p, 1-p)$ does not belong to \mathcal{R}_1 , so (ϕ, ψ) cannot belong to R_1 . Thus R_1 is the diagonal relation on Ω .

It is easy to see that the measurable transpose of \mathcal{R}_1 is taken to the classical transpose of R_1 . To verify compatibility with products, fix $\phi, \theta \in \Omega$; we must show that $(\phi, \theta) \in R_1 R_2$ if and only if $(p, r) \in \mathcal{R}_1 \mathcal{R}_2$ for all projections p and r such that $\phi(p) = \theta(r) = 1$. For the forward implication, let p and r be projections such that $\phi(p) = \theta(r) = 1$ and suppose that $(p, r) \notin \mathcal{R}_1 \mathcal{R}_2$. Let q' be the join of all projections p' such that $(p, p') \notin \mathcal{R}_1$ and let q'' be the join of all projections r' such that $(r', r) \notin \mathcal{R}_2$. Then $(p, q') \notin \mathcal{R}_1$ and $(q'', r) \notin \mathcal{R}_2$, and we must have $q' \vee q'' = 1$ as otherwise $1 - (q' \vee q'')$ would witness $(p, r) \in \mathcal{R}_1 \mathcal{R}_2$. Then for any $\psi \in \Omega$ we have either $\psi(q') = 1$ or $\psi(q'') = 1$, which implies that either $(\phi, \psi) \notin R_1$ or $(\psi, \theta) \notin R_2$. This shows that $(\phi, \theta) \notin R_1 R_2$, which completes the proof of the forward implication. For the reverse implication, suppose that $(p, r) \in \mathcal{R}_1 \mathcal{R}_2$ for all projections p and r such that $\phi(p) = \theta(r) = 1$. For any finite partition $\mathcal{S} = \{S_1, \dots, S_m\}$ of X we have $\phi(\chi_{S_i}) = \theta(\chi_{S_k}) = 1$ for precisely one choice of i and k . Since $(\chi_{S_i}, \chi_{S_k}) \in \mathcal{R}_1 \mathcal{R}_2$ there exists a nonzero projection q such that $(\chi_{S_i}, q') \in \mathcal{R}_1$ and $(q', \chi_{S_k}) \in \mathcal{R}_2$ for every $q' \leq q$. Choose a value of j such that $q\chi_{S_j} \neq 0$; then $(\chi_{S_i}, \chi_{S_j}) \in \mathcal{R}_1$ and $(\chi_{S_j}, \chi_{S_k}) \in \mathcal{R}_2$. Define $\psi_S : L^\infty(X, \mu) \rightarrow \mathbf{C}$ by

$$\psi_S(f) = \frac{1}{\mu(S_j)} \int_{S_j} f d\mu$$

and let ψ be a weak* cluster point of the net $\{\psi_S\}$. Then as in an earlier part of the proof we have $(\phi, \psi) \in R_1$ and $(\psi, \theta) \in R_2$, so $(\phi, \theta) \in R_1 R_2$. This completes the proof of the reverse implication. \square

It immediately follows that all of the notions introduced in Definition 1.8 reduce to their classical analogs under the conversion described above.

For our purposes there is no need to convert to pointwise relations, so we will not use Theorem 1.10. Nonetheless, it may be cited as evidence that measurable

relations are a reasonable generalization of pointwise relations to the measurable setting.

1.4. Basic results. In this section we present three basic tools for working with measurable relations. The first is easy but the other two are more substantive.

Proposition 1.11. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space and suppose $(p, q) \in \mathcal{R}$. Then there exist nonzero projections $p' \leq p$ and $q' \leq q$ such that*

$$(p', q'') \in \mathcal{R} \quad \text{and} \quad (p'', q') \in \mathcal{R}$$

for all nonzero projections $p'' \leq p'$ and $q'' \leq q'$.

Proof. Let r be the join of $\{r' \leq p : (r', q) \notin \mathcal{R}\}$ and let s be the join of $\{s' \leq q : (p, s') \notin \mathcal{R}\}$; then take $p' = p - r$ and $q' = q - s$. By the definition of measurable relations we must have $(r, q), (p, s) \notin \mathcal{R}$, so that $r \neq p$ and $s \neq q$. Therefore p' and q' are nonzero.

Now let $p'' \leq p'$ be nonzero. Then $p'' \not\leq r$, so $(p'', q) \in \mathcal{R}$. But $(p'', s) \notin \mathcal{R}$ because $(p, s) \notin \mathcal{R}$ and $p'' \leq p$; since $q = s \vee q'$ and $(p'', q) \in \mathcal{R}$ this implies $(p'', q') \in \mathcal{R}$. The analogous statement for nonzero $q'' \leq q'$ holds by symmetry. \square

Say that a projection p in $L^\infty(X, \mu)$ is an *atom* if there is no projection q with $0 < q < p$, and is *nonatomic* if it does not lie above any atoms. Also, say that p has *finite measure* if $\int p d\mu < \infty$.

Lemma 1.12. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space and suppose $(p, q) \in \mathcal{R}$ with p and q both nonatomic with finite measure. Say $p = \chi_S$ and $q = \chi_T$. Then there exists $a > 0$ such that any finite partitions $\{S_1, \dots, S_m\}$ and $\{T_1, \dots, T_n\}$ of S and T can be refined to partitions $\{S'_1, \dots, S'_{m'}\}$ and $\{T'_1, \dots, T'_{n'}\}$ with the properties that (1) we have*

$$2 \cdot \min_{i,j} \{\mu(S'_i), \mu(T'_j)\} \geq \max_{i,j} \{\mu(S'_i), \mu(T'_j)\}$$

and (2) after reordering, there exists k such that $\mu(S'_1 \cup \dots \cup S'_k) \geq a$ and $(\chi_{S'_l}, \chi_{T'_l}) \in \mathcal{R}$ for $1 \leq l \leq k$.

Proof. Fix partitions $\{S_1, \dots, S_m\}$ and $\{T_1, \dots, T_n\}$ of S and T . Let $b = \min\{\mu(S_i), \mu(T_j)\}$. Since p and q are nonatomic and have finite measure we can refine to partitions $\{S'_1, \dots, S'_{m'}\}$ and $\{T'_1, \dots, T'_{n'}\}$ such that $b \leq \mu(S'_i), \mu(T'_j) \leq 2b$ for all i and j (see [14], Theorem III.1.22). We then follow a greedy algorithm, pairing S 's and T 's that belong to the relation until no further pair can be found. Then after reordering there exists k such that $(\chi_{S'_l}, \chi_{T'_l}) \in \mathcal{R}$ for $1 \leq l \leq k$ and $(\chi_{S'_i}, \chi_{T'_j}) \notin \mathcal{R}$ for any $i, j > k$.

Let $a = \mu(S'_1 \cup \dots \cup S'_k)$. We have to show that there is a positive lower bound on a independent of the construction we just performed and of the original choice of partitions of S and T . Suppose not. Then for each n we can find some such construction with $a \leq 2^{-n}$. Let $U_n = S'_{k+1} \cup \dots \cup S'_{m'}$ and $V_n = T'_{k+1} \cup \dots \cup T'_{n'}$. Then $(\chi_{U_n}, \chi_{V_n}) \notin \mathcal{R}$ since $(\chi_{S'_i}, \chi_{T'_j}) \notin \mathcal{R}$ for any $i, j > k$. So for every N we have

$$\left(\bigvee_{n \geq N} \chi_{U_n}, \bigwedge_{n \geq N} \chi_{V_n} \right) \notin \mathcal{R},$$

and $p = \bigvee_{n \geq N} \chi_{U_n}$, so $(p, \bigwedge_{n \geq N} \chi_{V_n}) \notin \mathcal{R}$. Since $q = \bigvee_N \bigwedge_{n \geq N} \chi_{V_n}$ this implies $(p, q) \notin \mathcal{R}$, a contradiction. Therefore there is a positive lower bound on a , as claimed. \square

For $f \in L^\infty(X, \mu)$ let $M_f \in B(L^2(X, \mu))$ be the multiplication operator $M_f : g \mapsto fg$.

Theorem 1.13. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space and suppose $(p, q) \in \mathcal{R}$ with $p = \chi_S$ and $q = \chi_T$. Then there is a nonzero bounded operator $A : L^2(T, \mu|_T) \rightarrow L^2(S, \mu|_S)$ such that $f \geq 0$ implies $Af \geq 0$ and*

$$(p', q') \notin \mathcal{R} \quad \Rightarrow \quad M_{p'} A M_{q'} = 0$$

for $p' \leq p$ and $q' \leq q$.

Proof. Let r be the join of the atoms lying under p . If $(r, q) \in \mathcal{R}$ then there is an atom $\chi_{S'} \leq r$ such that $(\chi_{S'}, q) \in \mathcal{R}$. Applying Proposition 1.11, we can find a nonzero finite measure projection $q' \leq q$ such that $(\chi_{S'}, q'') \in \mathcal{R}$ for every $q'' \leq q'$. Writing $q' = \chi_{T'}$, we can then take A to be the operator

$$f \mapsto \left(\int_{T'} f d\mu \right) \cdot \chi_{S'}.$$

This has the desired properties. So we can assume $(r, q) \notin \mathcal{R}$, and replacing p with $p - r$ we may assume p is nonatomic. A similar argument reduces to the case where q is also nonatomic.

Since p is the join of the finite measure projections lying under p and the same is true of q , we can find finite measure projections $p' \leq p$ and $q' \leq q$ such that $(p', q') \in \mathcal{R}$. Replacing p with p' and q with q' , we may now suppose that p and q are both nonatomic with finite measure. We can therefore apply Lemma 1.12. Fix $a > 0$ as in this lemma and for any finite partitions $\mathcal{S} = \{S_1, \dots, S_m\}$ of S and $\mathcal{T} = \{T_1, \dots, T_n\}$ of T , let $\{S'_1, \dots, S'_{m'}\}$ and $\{T'_1, \dots, T'_{n'}\}$ be the refined partitions and k the integer provided by the lemma. Then define $A_{\mathcal{S}, \mathcal{T}} : L^2(T, \mu|_T) \rightarrow L^2(S, \mu|_S)$ by

$$A_{\mathcal{S}, \mathcal{T}} : f \mapsto \sum_{l=1}^k \frac{1}{\mu(T'_l)} \left(\int_{T'_l} f d\mu \right) \cdot \chi_{S'_l}.$$

The operator norm of $A_{\mathcal{S}, \mathcal{T}}$ is

$$\|A_{\mathcal{S}, \mathcal{T}}\| = \max_{1 \leq l \leq k} \sqrt{\frac{\mu(S'_l)}{\mu(T'_l)}} \leq \sqrt{2}$$

so the net $\{A_{\mathcal{S}, \mathcal{T}}\}$ is bounded. Let A be a weak operator cluster point of this net. We immediately have $f \geq 0 \Rightarrow Af \geq 0$ since this is true of every $A_{\mathcal{S}, \mathcal{T}}$. Also, $A \neq 0$ because

$$\langle A_{\mathcal{S}, \mathcal{T}}(\chi_T), \chi_S \rangle = \int_S A_{\mathcal{S}, \mathcal{T}}(\chi_T) d\mu = \mu(S'_1 \cup \dots \cup S'_k) \geq a$$

for all \mathcal{S}, \mathcal{T} , which implies that $\langle A(\chi_T), \chi_S \rangle \geq a$. Finally, let $S' \subseteq S$ and $T' \subseteq T$ satisfy $(\chi_{S'}, \chi_{T'}) \notin \mathcal{R}$. Then for any partitions \mathcal{S} and \mathcal{T} which respectively refine the partitions $\{S', S - S'\}$ and $\{T', T - T'\}$ we have $M_{\chi_{S'}} A_{\mathcal{S}, \mathcal{T}} M_{\chi_{T'}} = 0$ by construction (since no projection under $\chi_{S'}$ will be paired with a projection under $\chi_{T'}$). Taking the weak operator limit then shows that $M_{\chi_{S'}} A M_{\chi_{T'}} = 0$. \square

Our last result in this section is the most powerful. Its proof uses basic von Neumann algebra techniques.

Theorem 1.14. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space (X, μ) and suppose $(p, q) \in \mathcal{R}$. Then there is a finitely decomposable measure space (Y, ν) and a pair of unital weak* continuous *-homomorphisms $\pi_l, \pi_r : L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ such that $\pi_l(p) = \pi_r(q) = 1$ and*

$$(p', q') \notin \mathcal{R} \quad \Rightarrow \quad \pi_l(p')\pi_r(q') = 0.$$

Proof. Say $p = \chi_S$ and $q = \chi_T$ and let A be the operator provided by Theorem 1.13. As in the proof of Theorem 1.13 we may assume $\mu(S), \mu(T) < \infty$. Define a linear functional τ on the algebraic tensor product $\mathcal{A} = L^\infty(S, \mu|_S) \otimes L^\infty(T, \mu|_T)$ by

$$\tau(f \otimes g) = \int f A g d\mu|_S = \langle A g, \bar{f} \rangle$$

(and extending linearly). We claim that τ is positive in the sense that

$$\tau\left(\left(\sum f_i \otimes g_i\right)\left(\sum \bar{f}_i \otimes \bar{g}_i\right)\right) \geq 0$$

for any $\sum f_i \otimes g_i \in \mathcal{A}$. By continuity it is enough to check this when the f_i and g_i are simple. So fix partitions $\{S_1, \dots, S_m\}$ and $\{T_1, \dots, T_n\}$ of S and T and suppose the f_i and g_i are constant on each S_k and each T_l , respectively. For $1 \leq k \leq m$ and $1 \leq l \leq n$ let $a_{kl} = \langle A(\chi_{T_l}), \chi_{S_k} \rangle$ and observe that each $a_{kl} \geq 0$ since $f \geq 0 \Rightarrow Af \geq 0$.

Say $f_i = \sum b_{ik} \chi_{S_k}$ and $g_i = \sum c_{il} \chi_{T_l}$. We now compute

$$\begin{aligned} \tau\left(\left(\sum_i f_i \otimes g_i\right)\left(\sum_i \bar{f}_i \otimes \bar{g}_i\right)\right) &= \sum_{i,j} \langle A(g_i \bar{g}_j), \bar{f}_i f_j \rangle \\ &= \sum_{i,j,k,l} b_{ik} \bar{b}_{jk} c_{il} \bar{c}_{jl} a_{kl} \\ &= \sum_{k,l} a_{kl} \left| \sum_i b_{ik} c_{il} \right|^2 \\ &\geq 0. \end{aligned}$$

This establishes the claim.

Now define a pre-inner product on \mathcal{A} by setting

$$\langle f \otimes g, f' \otimes g' \rangle = \tau((f \otimes g)(\bar{f}' \otimes \bar{g}'))$$

(and extending linearly) and let H be the Hilbert space formed by factoring out null vectors and completing. Define representations $\pi_l, \pi_r : L^\infty(X, \mu) \rightarrow \mathcal{B}(H)$ by $\pi_l(h)(f \otimes g) = h|_S f \otimes g$ and $\pi_r(h)(f \otimes g) = f \otimes h|_T g$. The representation π_l is both well-defined and contractive by the calculation

$$\begin{aligned} \left\| \pi_l(h) \left(\sum_i f_i \otimes g_i \right) \right\|_H^2 &= \tau \left(\sum_{i,j} |h|_S|^2 f_i \bar{f}_j \otimes g_i \bar{g}_j \right) \\ &\leq \tau \left(\sum_{i,j} \|h\|_\infty^2 f_i \bar{f}_j \otimes g_i \bar{g}_j \right) \end{aligned}$$

$$\begin{aligned}
&= \|h\|_\infty^2 \cdot \tau \left(\sum_{i,j} f_i \bar{f}_j \otimes g_i \bar{g}_j \right) \\
&= \|h\|_\infty^2 \left\| \sum_i f_i \otimes g_i \right\|_H^2,
\end{aligned}$$

and a similar calculation shows the same for π_r . (The inequality that replaces $|h|_S|^2$ with $\|h\|_\infty^2$ follows from positivity of τ , or it can be proven first for simple f, g, h by a computation similar to the one used above to prove that τ is positive.) The sets $\pi_l(L^\infty(X, \mu))$ and $\pi_r(L^\infty(X, \mu))$ generate an abelian von Neumann algebra $\mathcal{M} \cong L^\infty(Y, \nu)$. If $(p', q') \notin \mathcal{R}$ then $M_{p'} A M_{q'} = 0$ implies that $\pi_l(p') \pi_r(q') = 0$ because

$$\begin{aligned}
\langle \pi_l(p') \pi_r(q')(f \otimes g), f' \otimes g' \rangle &= \tau((p'|_S f \otimes q'|_T g)(\bar{f}' \otimes \bar{g}')) \\
&= \langle A(q'|_T g \bar{g}'), p'|_S f f' \rangle \\
&= \langle (M_{p'} A M_{q'}) g \bar{g}', \bar{f} f' \rangle \\
&= 0
\end{aligned}$$

for all $f, f' \in L^\infty(S, \mu|_S)$ and $g, g' \in L^\infty(T, \mu|_T)$. All other properties of π_l and π_r are routinely verified. \square

1.5. Measurable metrics. Measurable metric spaces were introduced in [15] and have subsequently been studied in connection with derivations [18, 20, 19] and local Dirichlet forms [7, 8, 9]. Unfortunately, as was pointed out by Francis Hirsch, there is an error in one of the basic results, Lemma 6.1.6 of [18], which was used heavily in developing the general theory of these structures. Most of the resulting problems can be fixed without too much trouble — some of this has been done in [7] — but in some places the faulty lemma was really used in an essential way.

The machinery we developed in Section 1.4, particularly Theorem 1.14, can be used to quickly correct all of the problems stemming from the use of the erroneous lemma. We do this now.

We recall the basic definition:

Definition 1.15. ([18], Definition 6.1.3) Let (X, μ) be a finitely decomposable measure space and let \mathcal{P} be the set of nonzero projections in $L^\infty(X, \mu)$. A *measurable pseudometric* on (X, μ) is a function $\rho : \mathcal{P}^2 \rightarrow [0, \infty]$ such that

- (i) $\rho(p, p) = 0$
- (ii) $\rho(p, q) = \rho(q, p)$
- (iii) $\rho(\bigvee p_\lambda, \bigvee q_\kappa) = \inf_{\lambda, \kappa} \rho(p_\lambda, q_\kappa)$
- (iv) $\rho(p, r) \leq \sup_{q' \leq q} (\rho(p, q') + \rho(q', r))$

for all $p, q, r, p_\lambda, q_\kappa \in \mathcal{P}$. It is a *measurable metric* if for all disjoint p and q there exist nets $\{p_\lambda\}$ and $\{q_\lambda\}$ such that $p_\lambda \rightarrow p$ and $q_\lambda \rightarrow q$ weak* and $\rho(p_\lambda, q_\lambda) > 0$ for all λ .

If either p or q is (or both are) the zero projection then the appropriate convention is $\rho(p, q) = \infty$. (Note that in the measurable triangle inequality, property (iv), q' ranges over nonzero projections.)

Say that the *closure* of p is the complement of $\bigvee \{q : \rho(p, q) > 0\}$, and say that p is *closed* if it equals its closure. Equivalently, p is closed if for every nonzero projection q that is disjoint from p there exists a nonzero projection $q' \leq q$ such that $\rho(p, q') > 0$. Then ρ is a measurable metric if and only if the closed projections

generate $L^\infty(X, \mu)$ as a von Neumann algebra. This was actually the definition of measurable metric used in [18], and the equivalence with the condition given above follows from ([18], Theorem 6.2.10).

We begin with a simple observation connecting measurable metrics to measurable relations. Its proof is an easy verification.

Lemma 1.16. *Let (X, μ) be a finitely decomposable measure space, let ρ be a measurable pseudometric on X , and let $t > 0$. Then*

$$\mathcal{R}_t = \{(p, q) : \rho(p, q) < t\}$$

is a reflexive, symmetric measurable relation on X . We have $\mathcal{R}_s \mathcal{R}_t \subseteq \mathcal{R}_{s+t}$ for all $s, t > 0$.

Proof. The first assertion is straightforward. To verify the second, suppose $(p, r) \in \mathcal{R}_s \mathcal{R}_t$ and find a nonzero projection q such that $(p, q') \in \mathcal{R}_s$ and $(q', r) \in \mathcal{R}_t$ for every nonzero $q' \leq q$ (Proposition 1.5 (c)). Then $\rho(p, q) < s$, so let $\epsilon = (s - \rho(p, q))/2$ and define $q_1 = q(1 - \tilde{q})$ where

$$\tilde{q} = \bigvee \{q' : \rho(p, q') \geq \rho(p, q) + \epsilon\}.$$

Note that

$$\rho(p, q\tilde{q}) \geq \rho(p, \tilde{q}) \geq \rho(p, q) + \epsilon$$

so $q\tilde{q} \neq q$, i.e., q_1 is nonzero. We then have that $\rho(p, q') < \rho(p, q) + \epsilon = s - \epsilon$ for all $q' \leq q_1$, and we still have $\rho(q', r) < t$ for all $q' \leq q_1$ since $q_1 \leq q$. Then the measurable triangle inequality (Definition 1.15 (iv)) implies $\rho(p, r) \leq s + t - \epsilon$, so that $(p, r) \in \mathcal{R}_{s+t}$. \square

It follows that using the method of Theorem 1.10 we can convert measurable pseudometrics on (X, μ) into pointwise pseudometrics on the carrier space of $L^\infty(X, \mu)$. This result is an improved version of Theorem 6.3.9 of [18]. We retain the notation of Section 1.3.

Theorem 1.17. *Let ρ be a measurable pseudometric on a finitely decomposable measure space (X, μ) . Then $d_\rho : \Omega^2 \rightarrow [0, \infty]$ defined by*

$$d_\rho(\phi, \psi) = \sup\{\rho(p, q) : \phi(p) = \psi(q) = 1\}$$

is a pseudometric on Ω , and we have

$$\rho(p, q) = \inf\{d_\rho(\phi, \psi) : \phi(p) = \psi(q) = 1\}$$

for any nonzero projections p and q in $L^\infty(X, \mu)$.

Proof. For any $\phi \in \Omega$ and any projections $p, q \in L^\infty(X, \mu)$, if $\phi(p) = \phi(q) = 1$ then $\phi(pq) = 1$ and hence $pq \neq 0$; by Definition 1.15 (i) and (iii) this implies $\rho(p, q) = 0$. This shows that $d_\rho(\phi, \phi) = 0$ for all $\phi \in \Omega$. Symmetry of d_ρ follows immediately from symmetry of ρ . For the triangle inequality, let $\phi, \psi, \theta \in \Omega$ and fix projections $p, r \in L^\infty(X, \mu)$ such that $\phi(p) = \theta(r) = 1$. We may assume that $d_\rho(\phi, \psi), d_\rho(\psi, \theta) < \infty$. Let $\epsilon > 0$, let p' be the join of the projections p'' such that $\rho(p, p'') \geq d_\rho(\phi, \psi) + \epsilon$, and let r' be the join of the projections r'' such that $\rho(r'', r) \geq d_\rho(\psi, \theta) + \epsilon$. Then $\rho(p, p') \geq d_\rho(\phi, \psi) + \epsilon$, and since $\phi(p) = 1$ we must therefore have $\psi(p') = 0$ by the definition of $d_\rho(\phi, \psi)$. Similarly $\psi(r') = 0$, so letting $q' = (1 - p')(1 - r')$ we must have $\psi(q') = \psi(1 - p')\psi(1 - r') = 1$. Therefore $q' \neq 0$, so

$$\rho(p, r) \leq \sup_{q'' \leq q'} (\rho(p, q'') + \rho(q'', r)) \leq d_\rho(\phi, \psi) + d_\rho(\psi, \theta) + 2\epsilon$$

since any $q'' \leq q'$ is disjoint from both p' and r' . Taking $\epsilon \rightarrow 0$ and then taking the supremum over p and r yields $d_\rho(\phi, \theta) \leq d_\rho(\phi, \psi) + d_\rho(\psi, \theta)$. So d_ρ is a pseudometric on Ω .

Let p and q be nonzero projections in $L^\infty(X, \mu)$. It is immediate from the definition of $d_\rho(\phi, \psi)$ that $\rho(p, q) \leq d_\rho(\phi, \psi)$ for any $\phi, \psi \in \Omega$ satisfying $\phi(p) = \psi(q) = 1$. Conversely, given $\epsilon > 0$ we need to find $\phi, \psi \in \Omega$ such that $\phi(p) = \psi(q) = 1$ and $d_\rho(\phi, \psi) \leq \rho(p, q) + \epsilon$. We may assume $\rho(p, q) < \infty$. Use Lemma 1.16 and Theorem 1.14 to find a finitely decomposable measure space (Y, ν) and a pair of unital weak* continuous *-homomorphisms $\pi_l, \pi_r : L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ such that $\pi_l(p) = \pi_r(q) = 1$ and $\pi_l(p')\pi_r(q') = 0$ for any nonzero projections p' and q' with $\rho(p', q') \geq \rho(p, q) + \epsilon$. Let ϕ be any nonzero homomorphism from $L^\infty(Y, \nu)$ to \mathbf{C} ; then $\phi \circ \pi_l$ and $\phi \circ \pi_r$ belong to Ω and satisfy $\phi \circ \pi_l(p) = \phi \circ \pi_r(q) = 1$. Also, if p' and q' are any nonzero projections in $L^\infty(X, \mu)$ such that $\phi \circ \pi_l(p') = \phi \circ \pi_r(q') = 1$ then $\phi(\pi_l(p')\pi_r(q')) = 1$, so $\pi_l(p')\pi_r(q') \neq 0$, and hence $\rho(p', q') < \rho(p, q) + \epsilon$. Thus $d_\rho(\phi \circ \pi_l, \phi \circ \pi_r) \leq \rho(p, q) + \epsilon$, as desired. \square

Next, we define measurable Lipschitz numbers. Recall that the *essential range* of a function $f \in L^\infty(X, \mu)$ is the set of all $a \in \mathbf{C}$ such that $f^{-1}(U)$ has positive measure for every open neighborhood U of a . Equivalently, it is the spectrum of the operator $M_f \in \mathcal{B}(L^2(X, \mu))$. If $p \in L^\infty(X, \mu)$ is a projection then we denote the essential range of $f|_{\text{supp}(p)}$ by $\text{ran}_p(f)$.

Definition 1.18. ([18], Definition 6.2.1) Let (X, μ) be a finitely decomposable measure space and let ρ be a measurable pseudometric on X . The *Lipschitz number* of $f \in L^\infty(X, \mu)$ is the quantity

$$L(f) = \sup \left\{ \frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p, q)} \right\},$$

where the supremum is taken over all nonzero projections $p, q \in L^\infty(X, \mu)$ and we use the convention $\frac{0}{0} = 0$. Here d is the usual (minimum) distance between compact subsets of \mathbf{C} . We call L the *Lipschitz gauge* associated to ρ and we define $\text{Lip}(X, \mu) = \{f \in L^\infty(X, \mu) : L(f) < \infty\}$.

Now we introduce the key tool for studying Lipschitz numbers.

Definition 1.19. ([18], Definition 6.3.1) Let (X, μ) be a finitely decomposable measure space and let ρ be a measurable pseudometric on X . For any nonzero projections $p, q \in L^\infty(X, \mu)$ and any $\epsilon > 0$, let $\mathcal{R}_{p, q, \epsilon}$ be the measurable relation defined in Lemma 1.16 with $t = \rho(p, q) + \epsilon$, so that $(p, q) \in \mathcal{R}_{p, q, \epsilon}$, and find $\pi_l^{p, q, \epsilon}$, $\pi_r^{p, q, \epsilon}$, and $(Y_{p, q, \epsilon}, \nu_{p, q, \epsilon})$ as in Theorem 1.14. Then let $Y = \coprod Y_{p, q, \epsilon}$ and $\nu = \coprod \nu_{p, q, \epsilon}$, and for $f \in L^\infty(X, \mu)$ define $\Phi(f)$ on Y by

$$\Phi(f)|_{Y_{p, q, \epsilon}} = \frac{\pi_l^{p, q, \epsilon}(f) - \pi_r^{p, q, \epsilon}(f)}{\rho(p, q) + \epsilon}.$$

Also define $\pi_l, \pi_r : L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ by $\pi_l = \bigoplus \pi_l^{p, q, \epsilon}$ and $\pi_r = \bigoplus \pi_r^{p, q, \epsilon}$. We call Φ a *measurable de Leeuw map*.

Theorem 1.20. ([18], Theorem 6.3.2) Let (X, μ) be a finitely decomposable measure space, let ρ be a measurable pseudometric on X , and let Φ be a measurable de Leeuw map.

- (a) For all $f \in L^\infty(X, \mu)$ we have $L(f) = \|\Phi f\|_\infty$.
- (b) Φ is linear and we have $\Phi(fg) = \pi_l(f)\Phi(g) + \Phi(f)\pi_r(g)$ for all $f, g \in L^\infty(X, \mu)$.

(c) The graph of $\Phi : \text{Lip}(X, \mu) \rightarrow L^\infty(Y, \nu)$ is weak* closed in $L^\infty(X \amalg Y)$.

Proof. (a) Let $p, q \in L^\infty(X, \mu)$ be nonzero projections. Since $\pi_l^{p,q,\epsilon}(p) = \pi_r^{p,q,\epsilon}(q) = 1_{Y_{p,q,\epsilon}}$, the essential ranges of $\pi_l^{p,q,\epsilon}(f)$ and $\pi_r^{p,q,\epsilon}(f)$ are respectively contained in $\text{ran}_p(f)$ and $\text{ran}_q(f)$, so

$$\|\pi_l^{p,q,\epsilon}(f) - \pi_r^{p,q,\epsilon}(f)\|_\infty \geq d(\text{ran}_p(f), \text{ran}_q(f)).$$

Thus

$$\|\Phi(f)\|_\infty \geq \frac{d(\text{ran}_p(f), \text{ran}_q(f))}{\rho(p, q) + \epsilon};$$

taking $\epsilon \rightarrow 0$ and the supremum over p and q then yields $\|\Phi(f)\|_\infty \geq L(f)$.

To verify the reverse inequality, fix p, q , and ϵ ; we will show that $\|\Phi(f)|_{Y_{p,q,\epsilon}}\|_\infty \leq L(f)$. We may assume that $\|\pi_l^{p,q,\epsilon}(f) - \pi_r^{p,q,\epsilon}(f)\|_\infty > 0$. Let $0 < \delta < \|\pi_l^{p,q,\epsilon}(f) - \pi_r^{p,q,\epsilon}(f)\|_\infty / 2$ and partition p and q as $p = \sum_1^m p_i$, $q = \sum_1^n q_j$ so that $\text{ran}_{p_i}(f)$ and $\text{ran}_{q_j}(f)$ have diameter at most δ for all i and j . Then for some choice of i and j we must have $\pi_l^{p,q,\epsilon}(p_i)\pi_r^{p,q,\epsilon}(q_j) \neq 0$ and

$$d(\text{ran}_{p_i}(f), \text{ran}_{q_j}(f)) \geq \|\pi_l^{p,q,\epsilon}(f) - \pi_r^{p,q,\epsilon}(f)\|_\infty - 2\delta.$$

Then $\rho(p_i, q_j) \leq \rho(p, q) + \epsilon$ and hence

$$\frac{\|\pi_l^{p,q,\epsilon}(f) - \pi_r^{p,q,\epsilon}(f)\|_\infty}{\rho(p, q) + \epsilon} \leq \frac{d(\text{ran}_{p_i}(f), \text{ran}_{q_j}(f)) + 2\delta}{\rho(p_i, q_j)}.$$

If $\rho(p_i, q_j) > 0$ then taking $\delta \rightarrow 0$ shows that $\|\Phi(f)|_{Y_{p,q,\epsilon}}\|_\infty \leq L(f)$; if $\rho(p_i, q_j) = 0$ then $L(f) = \infty$ since $d(\text{ran}_{p_i}(f), \text{ran}_{q_j}(f)) > 0$, so again $\|\Phi(f)|_{Y_{p,q,\epsilon}}\|_\infty \leq L(f)$. We conclude that $\|\Phi(f)\|_\infty \leq L(f)$.

(b) This is trivially verified on each $Y_{p,q,\epsilon}$ separately.

(c) Let $\{f_\lambda\}$ be a net in $\text{Lip}(X, \mu)$ and suppose $f_\lambda \oplus \Phi(f_\lambda) \rightarrow f \oplus g$ weak* in $L^\infty(X \amalg Y)$. Restricting to each $Y_{p,q,\epsilon}$ shows that $g = \Phi(f)$. Then $L(f) = \|g\|_\infty < \infty$ by part (a) so $f \in \text{Lip}(X, \mu)$, and we conclude that the graph of Φ is weak* closed. \square

Corollary 1.21. ([18], Lemma 6.2.6 and Theorem 6.2.7) Let (X, μ) be a finitely decomposable measure space and let ρ be a measurable pseudometric on X .

(a) $L(af) = |a| \cdot L(f)$, $L(\bar{f}) = L(f)$, $L(f + g) \leq L(f) + L(g)$, and $L(fg) \leq \|f\|_\infty L(g) + \|g\|_\infty L(f)$ for all $f, g \in \text{Lip}(X, \mu)$ and $a \in \mathbb{C}$.

(b) If $\{f_\lambda\} \subseteq L^\infty(X, \mu)$ is a net that converges weak* to $f \in L^\infty(X, \mu)$ then $L(f) \leq \sup L(f_\lambda)$.

(c) $\text{Lip}(X, \mu)$ is a self-adjoint unital subalgebra of $L^\infty(X, \mu)$. It is a dual Banach space for the norm $\|f\|_L = \max\{\|f\|_\infty, L(f)\}$.

(d) The real part of the unit ball of $\text{Lip}(X, \mu)$ is a complete sublattice of the real part of the unit ball of $L^\infty(X, \mu)$.

Proof. (a) This follows easily from Theorem 1.20 (a) and (b).

(b) If $f_\lambda \rightarrow f$ weak* then $\Phi(f_\lambda)|_{Y_{p,q,\epsilon}} \rightarrow \Phi(f)|_{Y_{p,q,\epsilon}}$ weak* for each p, q, ϵ . It follows that

$$L(f) = \|\Phi(f)\|_\infty \leq \sup \|\Phi(f_\lambda)\|_\infty = \sup L(f_\lambda).$$

(c) The first assertion follows immediately from part (a) and the second assertion follows from Theorem 1.20 (c) because $\|f\|_L = \|f \oplus \Phi(f)\|_\infty$, so that $\text{Lip}(X, \mu)$ equipped with this norm is isometric to the graph of Φ .

(d) If $f, g \in \text{Lip}(X, \mu)$ are real-valued then $L(f \vee g) \leq \max\{L(f), L(g)\}$ because

$$d(\text{ran}_p(f \vee g), \text{ran}_q(f \vee g)) \leq \max\{d(\text{ran}_p(f), \text{ran}_q(f)), d(\text{ran}_p(g), \text{ran}_q(g))\}$$

for all p and q , and $L(f \wedge g) \leq \max\{L(f), L(g)\}$ similarly. So the real part of the unit ball of $\text{Lip}(X, \mu)$ is a sublattice of the real part of the unit ball of $L^\infty(X, \mu)$, and it is then a complete sublattice by part (b) since $\bigvee f_\lambda$ and $\bigwedge f_\lambda$ are respectively the weak* limits of the net of finite joins of the f_λ and the net of finite meets of the f_λ . \square

We include one more fundamental result.

Lemma 1.22. *Let (X, μ) be a finitely decomposable measure space and let ρ be a measurable pseudometric on X . Let $r \in L^\infty(X, \mu)$ be a nonzero projection and let $c > 0$. Then the function*

$$\bigvee \min\{\rho(p, r), c\} \cdot p,$$

taking the join in $L^\infty(X, \mu)$ over all nonzero projections p , has Lipschitz number at most 1.

Proof. Let f be this join, let p and q be nonzero projections, and let $\epsilon > 0$; we must show that $d(\text{ran}_p(f), \text{ran}_q(f)) \leq \rho(p, q) + \epsilon$. Let \mathcal{R} be the measurable relation defined in Lemma 1.16 with $t = \rho(p, q) + \epsilon$ and let p' and q' be the projections provided by Lemma 1.11. Find $p'' \leq p'$ such that $\text{ran}_{p''}(f)$ has diameter at most ϵ , then apply Lemma 1.11 to the pair (p'', q') , then do the same thing with p 's and q 's reversed. The result is a pair of nonzero projections $p_1 \leq p$ and $q_1 \leq q$ such that $\text{ran}_{p_1}(f)$ and $\text{ran}_{q_1}(f)$ both have diameter at most ϵ and $\rho(p_1, q_2), \rho(p_2, q_1) < \rho(p, q) + \epsilon$ for every nonzero $p_2 \leq p_1, q_2 \leq q_1$.

Let $a = \rho(p_1, r)$ and $b = \rho(q_1, r)$. We may assume $b \leq a \leq c$. Now apply Lemmas 1.16 and 1.11 to q_1 and r to find $q_2 \leq q_1$ such that $\rho(q_2, r) \leq \rho(q_1, r) + \epsilon$ for all $q_3 \leq q_2$. Then by the measurable triangle inequality we have

$$a = \rho(p_1, r) \leq \sup_{q_3 \leq q_2} (\rho(p_1, q_3) + \rho(q_3, r)) \leq \rho(p, q) + b + 2\epsilon.$$

We claim that $\text{ran}_{p_1}(f) \subseteq [a, a + \epsilon]$ and $\text{ran}_{q_2}(f) \subseteq [b, b + \epsilon]$. This is because, first, it is immediate from the definition of f that $f \geq ap_1$, and second, Lemmas 1.16 and 1.11 guarantee, for any $\delta > 0$, the existence of a nonzero projection $p_2 \leq p_1$ such that any nonzero projection under p_2 has distance at most $a + \delta$ to r , so that $fp_2 \leq (a + \delta)p_2$. This shows that $a \in \text{ran}_{p_1}(f)$, and since $\text{ran}_{p_1}(f)$ has diameter at most ϵ we conclude that $\text{ran}_{p_1}(f) \subseteq [a, a + \epsilon]$. The same argument applies to $\text{ran}_{q_1}(f)$. We therefore have

$$d(\text{ran}_p(f), \text{ran}_q(f)) \leq a - b \leq \rho(p, q) + 2\epsilon,$$

which is enough. \square

The join in Lemma 1.22 is a measurable version of the pointwise distance function $x \mapsto \min\{d(x, S), c\}$. Note that as long as there exists a nonzero projection p with $0 < \rho(p, r) < \infty$ the reverse inequality is easy, i.e., the Lipschitz number of the join is exactly 1.

2. QUANTUM RELATIONS

We now proceed to our definition of a quantum relation on a von Neumann algebra in terms of a bimodule over the commutant. The rough intuition is that the bimodule consists of the operators that only connect pairs of points that belong to the relation.

2.1. Basic definitions. Let H be a complex Hilbert space, not necessarily separable. Recall that the *weak** (or *σ -weak*) topology on $\mathcal{B}(H)$ is the weak topology arising from the pairing $\langle A, B \rangle \mapsto \text{tr}(AB)$ of $\mathcal{B}(H)$ with the trace class operators $\mathcal{TC}(H)$; that is, it is the weakest topology that makes the map $A \mapsto \text{tr}(AB)$ continuous for all $B \in \mathcal{TC}(H)$. The weak* topology is finer than the weak operator topology but the two agree on bounded sets. A *dual operator space* is a weak* closed subspace \mathcal{V} of $\mathcal{B}(H)$; it is a *W*-bimodule* over a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ if $\mathcal{M}\mathcal{V}\mathcal{M} \subseteq \mathcal{V}$.

We will refer to [14] for standard facts about von Neumann algebras.

Definition 2.1. A *quantum relation* on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ is a W*-bimodule over its commutant \mathcal{M}' , i.e., it is a weak* closed subspace $\mathcal{V} \subseteq \mathcal{B}(H)$ satisfying $\mathcal{M}'\mathcal{V}\mathcal{M}' \subseteq \mathcal{V}$.

The generalization to a quantum relation on a pair of von Neumann algebras $\mathcal{M} \subseteq \mathcal{B}(H)$ and $\mathcal{N} \subseteq \mathcal{B}(K)$ would be: a weak* closed subspace $\mathcal{V} \subseteq \mathcal{B}(H \oplus K)$ satisfying $\mathcal{M}'\mathcal{V}\mathcal{N}' \subseteq \mathcal{V}$. We need not develop this more general notion separately since quantum relations on \mathcal{M} and \mathcal{N} can be identified with quantum relations on the direct sum $\mathcal{M} \oplus \mathcal{N} \subseteq \mathcal{B}(H \oplus K)$ satisfying $\mathcal{V} = I_H \mathcal{V} I_K$ (i.e., that live in the (H, K) corner of $\mathcal{B}(H \oplus K)$).

As we noted above, the intuition is that \mathcal{V} consists of the operators that only connect pairs of points that belong to the relation. In the atomic abelian case this is exactly right. Recall the notations $e_x = \chi_{\{x\}}$ and $M_f : g \mapsto fg$. Also let V_{xy} be the rank one operator $V_{xy} : g \mapsto \langle g, e_y \rangle e_x$ on $l^2(X)$.

Proposition 2.2. *Let X be a set and let $\mathcal{M} \cong l^\infty(X)$ be the von Neumann algebra of bounded multiplication operators on $l^2(X)$. If R is a relation on X then*

$$\begin{aligned} \mathcal{V}_R &= \{A \in \mathcal{B}(l^2(X)) : (x, y) \notin R \Rightarrow \langle Ae_y, e_x \rangle = 0\} \\ &= \overline{\text{span}}^{wk*} \{V_{xy} : (x, y) \in R\} \end{aligned}$$

is a quantum relation on \mathcal{M} ; conversely, if \mathcal{V} is a quantum relation on \mathcal{M} then

$$R_{\mathcal{V}} = \{(x, y) \in X^2 : \langle Ae_y, e_x \rangle \neq 0 \text{ for some } A \in \mathcal{V}\}$$

is a relation on X . The two constructions are inverse to each other.

Proof. Note first that $\mathcal{M} = \mathcal{M}'$ in this case. Let $R \subseteq X^2$ and define \mathcal{V}_R to be the set of operators A such that $\langle Ae_y, e_x \rangle = 0$ for all $(x, y) \notin R$. Then it is clear that \mathcal{V}_R is a linear subspace of $\mathcal{B}(l^2(X))$. Also, \mathcal{V}_R is weak operator closed and therefore weak* closed. Finally, if M_f and M_g are any two multiplication operators in \mathcal{M} then

$$\langle Ae_y, e_x \rangle = 0 \Rightarrow \langle M_f A M_g e_y, e_x \rangle = f(x)g(y)\langle Ae_y, e_x \rangle = 0,$$

which shows that $A \in \mathcal{V}_R \Rightarrow M_f A M_g \in \mathcal{V}_R$. So \mathcal{V}_R is a W*-bimodule over $\mathcal{M}' = \mathcal{M}$.

We verify that $\mathcal{V}_R = \overline{\text{span}}^{wk*} \{V_{xy} : (x, y) \in R\}$. We have $(x, y) \in R \Rightarrow V_{xy} \in \mathcal{V}_R$ because $\langle V_{xy} e_{y'}, e_{x'} \rangle \neq 0$ only if $x = x'$ and $y = y'$; since \mathcal{V}_R is a weak* closed

subspace of $\mathcal{B}(l^2(X))$ this proves the inclusion \supseteq . For the reverse inclusion let $A \in \mathcal{V}_R$ and for any finite subset $F \subseteq X$ let $P_F \in \mathcal{B}(l^2(X))$ be the orthogonal projection onto $\text{span}\{e_x : x \in F\} \subseteq l^2(X)$. Then $P_F A P_F \rightarrow A$ boundedly weak operator and hence weak*, so it will suffice to show that each $P_F A P_F$ belongs to $\text{span}\{V_{xy} : (x, y) \in R\}$. But $P_F A P_F$ is a linear combination of operators of the form $M_{e_x} A M_{e_y}$ for $x, y \in X$, which are scalar multiples of the operators V_{xy} . Moreover,

$$M_{e_x} A M_{e_y} \neq 0 \Rightarrow \langle A e_y, e_x \rangle \neq 0 \Rightarrow (x, y) \in R.$$

So $P_F A P_F$ is a linear combination of operators V_{xy} with $(x, y) \in R$, as desired. This proves the inclusion \subseteq .

The second assertion of the proposition is trivial: $R_{\mathcal{V}}$ is a subset of X^2 directly from its definition.

Now let R be a relation, let $\mathcal{V} = \mathcal{V}_R$, and let $\tilde{R} = R_{\mathcal{V}}$. It is immediate that $\tilde{R} \subseteq R$. Conversely, let $(x, y) \in R$; then V_{xy} belongs to \mathcal{V} and satisfies $\langle V_{xy} e_y, e_x \rangle \neq 0$, so $(x, y) \in \tilde{R}$. Thus $R = \tilde{R}$.

Finally, let \mathcal{V} be a quantum relation, let $R = R_{\mathcal{V}}$, and let $\tilde{\mathcal{V}} = \mathcal{V}_R$. It is immediate that $\mathcal{V} \subseteq \tilde{\mathcal{V}}$. For the reverse inclusion it will suffice to show that $V_{xy} \in \mathcal{V}$ for all $(x, y) \in R$. But if $(x, y) \in R$ then we must have $\langle B e_y, e_x \rangle \neq 0$ for some $B \in \mathcal{V}$. Then $M_{e_x} B M_{e_y}$ is a nonzero scalar multiple of V_{xy} , and it belongs to \mathcal{V} since \mathcal{V} is a bimodule over \mathcal{M} . So $V_{xy} \in \mathcal{V}$, as desired. \square

2.2. Constructions with quantum relations. Next we consider basic constructions that can be performed with quantum relations. The following proposition is trivial.

Proposition 2.3. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra.*

- (a) *The commutant \mathcal{M}' is a quantum relation on \mathcal{M} .*
- (b) *If \mathcal{V} is a quantum relation on \mathcal{M} then so is $\mathcal{V}^* = \{A^* : A \in \mathcal{V}\}$.*
- (c) *If \mathcal{V} and \mathcal{W} are quantum relations on \mathcal{M} then so is the weak* closure of their algebraic product.*
- (d) *The intersection of any family of quantum relations on \mathcal{M} is a quantum relation on \mathcal{M} .*
- (e) *The weak* closed sum of any family of quantum relations on \mathcal{M} is a quantum relation on \mathcal{M} .*

This justifies the following definition.

Definition 2.4. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra.

- (a) The *diagonal quantum relation* on \mathcal{M} is the quantum relation $\mathcal{V} = \mathcal{M}'$.
- (b) The *transpose* of a quantum relation \mathcal{V} on \mathcal{M} is the quantum relation \mathcal{V}^* .
- (c) The *product* of two quantum relations \mathcal{V} and \mathcal{W} on \mathcal{M} is the weak* closure of their algebraic product.
- (d) A quantum relation \mathcal{V} on \mathcal{M} is
 - (i) *reflexive* if $\mathcal{M}' \subseteq \mathcal{V}$
 - (ii) *symmetric* if $\mathcal{V}^* = \mathcal{V}$
 - (iii) *antisymmetric* if $\mathcal{V} \cap \mathcal{V}^* \subseteq \mathcal{M}'$
 - (iv) *transitive* if $\mathcal{V}^2 \subseteq \mathcal{V}$.

We immediately note that Proposition 2.2 reduces the preceding notions to the classical ones in the atomic abelian case.

Proposition 2.5. *Let X be a set and let $\mathcal{M} \cong l^\infty(X)$ be the von Neumann algebra of bounded multiplication operators on $l^2(X)$. Also let R_1, R_2 , and R_3 be relations on X and let $\mathcal{V}_i = \mathcal{V}_{R_i}$ ($i = 1, 2, 3$) be the corresponding quantum relations on \mathcal{M} as in Proposition 2.2. Then*

- (a) $R_1 \subseteq R_2 \Leftrightarrow \mathcal{V}_1 \subseteq \mathcal{V}_2$
- (b) R_1 is the diagonal relation $\Leftrightarrow \mathcal{V}_1$ is the diagonal quantum relation
- (c) R_1 is the transpose of $R_2 \Leftrightarrow \mathcal{V}_1$ is the transpose of \mathcal{V}_2
- (d) R_3 is the product of R_1 and $R_2 \Leftrightarrow \mathcal{V}_3$ is the product of \mathcal{V}_1 and \mathcal{V}_2 .

The proof of this proposition is straightforward.

Using Definition 2.4 we can define quantum versions of equivalence relations, preorders, partial orders, and graphs.

Definition 2.6. Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra.

- (a) A *quantum equivalence relation* on \mathcal{M} is a reflexive, symmetric, transitive quantum relation on \mathcal{M} . That is, it is a von Neumann algebra that contains \mathcal{M}' .
- (b) A *quantum preorder* on \mathcal{M} is a reflexive, transitive quantum relation on \mathcal{M} . That is, it is a weak* closed operator algebra that contains \mathcal{M}' .
- (c) A *quantum partial order* on \mathcal{M} is a reflexive, antisymmetric, transitive quantum relation on \mathcal{M} . That is, it is a weak* closed operator algebra \mathcal{A} such that $\mathcal{A} \cap \mathcal{A}^* = \mathcal{M}'$.
- (d) A *quantum graph* on \mathcal{M} is a reflexive, symmetric quantum relation on \mathcal{M} . That is, it is a weak* closed operator system that is a bimodule over \mathcal{M}' .

Note that by the double commutant theorem, von Neumann algebras containing \mathcal{M}' correspond to von Neumann algebras contained in \mathcal{M} . So quantum equivalence relations on \mathcal{M} correspond to von Neumann subalgebras of \mathcal{M} . This is the expected definition.

If \mathcal{V} is a quantum preorder on \mathcal{M} then $\mathcal{V} \cap \mathcal{V}^*$ is a quantum equivalence relation on \mathcal{M} , i.e., $\mathcal{V} \cap \mathcal{V}^*$ is the commutant of some von Neumann subalgebra $\mathcal{M}_0 \subseteq \mathcal{M}$. Then \mathcal{V} is a quantum partial order on \mathcal{M}_0 . Passing from \mathcal{M} to \mathcal{M}_0 is the quantum version of factoring out equivalent elements to turn a preorder into a partial order.

As we noted following Definition 1.8, a graph can classically be encoded as a reflexive, symmetric relation. This justifies our definition of a quantum graph.

Definition 2.6 becomes especially simple when $\mathcal{M} = \mathcal{B}(H)$; in that case $\mathcal{M}' = \mathcal{C}I$, so that

- a quantum relation on $\mathcal{B}(H)$ is a dual operator space in $\mathcal{B}(H)$;
- a quantum equivalence relation on $\mathcal{B}(H)$ is a von Neumann algebra in $\mathcal{B}(H)$;
- a quantum preorder on $\mathcal{B}(H)$ is a weak* closed unital operator algebra in $\mathcal{B}(H)$;
- a quantum partial order on $\mathcal{B}(H)$ is a weak* closed operator algebra \mathcal{A} in $\mathcal{B}(H)$ satisfying $\mathcal{A} \cap \mathcal{A}^* = \mathcal{C}I$;
- a quantum graph on $\mathcal{B}(H)$ is a dual operator system in $\mathcal{B}(H)$.

For finite dimensional H , this definition of quantum graph was proposed in [4].

2.3. Basic results. Next we show that although the definition of a quantum relation is framed in terms of a particular representation, the notion is in fact representation independent. This is slightly surprising because the W^* -bimodules over \mathcal{M} do vary with the representation of \mathcal{M} : if we add multiplicity to a representation

(i.e., tensor with the identity on a nontrivial Hilbert space) the set of bimodules over \mathcal{M} that are contained in $\mathcal{B}(H)$ grows. But the commutant also grows, in such a way that the set of bimodules over \mathcal{M}' does not essentially change.

Theorem 2.7. *Let H_1 and H_2 be Hilbert spaces and let $\mathcal{M}_1 \subseteq \mathcal{B}(H_1)$ and $\mathcal{M}_2 \subseteq \mathcal{B}(H_2)$ be isomorphic von Neumann algebras. Then there is a 1-1 correspondence between the quantum relations on \mathcal{M}_1 and the quantum relations on \mathcal{M}_2 which respects the conditions $\mathcal{V} \subseteq \mathcal{W}$, $\mathcal{V} = \mathcal{M}'$, $\mathcal{V}^* = \mathcal{W}$, and $\mathcal{U}\mathcal{V} = \mathcal{W}$.*

Proof. Let K be any nonzero Hilbert space. Then

$$\mathcal{M}_i \cong I_K \otimes \mathcal{M}_i \subseteq \mathcal{B}(K \otimes H_i)$$

($i = 1, 2$). If the cardinality of K is large enough then the representations of $I_K \otimes \mathcal{M}_1$ and $I_K \otimes \mathcal{M}_2$ in $\mathcal{B}(K \otimes H_1)$ and $\mathcal{B}(K \otimes H_2)$, respectively, are spatially equivalent ([14], Theorem IV.5.5). So it is sufficient to consider the case where $H_2 = K \otimes H_1$ and $\mathcal{M}_2 = I_K \otimes \mathcal{M}_1$. We then have $\mathcal{M}'_2 = \mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$ ([14], Theorem IV.5.9).

Given a W^* -bimodule $\mathcal{V} \subseteq \mathcal{B}(H_1)$ over \mathcal{M}'_1 , let $\mathcal{B}(K) \overline{\otimes} \mathcal{V} \subseteq \mathcal{B}(K \otimes H_1)$ denote the normal spatial tensor product, i.e., the weak* closure of the algebraic tensor product in $\mathcal{B}(K \otimes H_1)$. It is clear that the map $\mathcal{V} \mapsto \mathcal{B}(K) \overline{\otimes} \mathcal{V}$ respects the conditions listed in the statement of the theorem. We must show that (1) $\mathcal{B}(K) \overline{\otimes} \mathcal{V}$ is a W^* -bimodule over $\mathcal{M}'_2 = \mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$; (2) if \mathcal{W} is a distinct W^* -bimodule over \mathcal{M}'_1 then $\mathcal{B}(K) \overline{\otimes} \mathcal{V} \neq \mathcal{B}(K) \overline{\otimes} \mathcal{W}$; and (3) every W^* -bimodule over $\mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$ is of the form $\mathcal{B}(K) \overline{\otimes} \mathcal{V}$ for some W^* -bimodule \mathcal{V} over \mathcal{M}'_1 .

It is clear that $\mathcal{B}(K) \overline{\otimes} \mathcal{V}$ is a weak* closed operator space in $\mathcal{B}(K \otimes H_1)$. Now \mathcal{V} is a bimodule over \mathcal{M}'_1 , so it is immediate that the algebraic tensor product $\mathcal{B}(K) \otimes \mathcal{V}$ is a bimodule over the algebraic tensor product $\mathcal{B}(K) \otimes \mathcal{M}'_1$. Taking weak* limits then shows that $\mathcal{B}(K) \overline{\otimes} \mathcal{V}$ is a bimodule over $\mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$. This verifies (1).

To verify (2), let P be a rank 1 projection in $\mathcal{B}(K)$. We claim that

$$(P \otimes I_{H_1})(\mathcal{B}(K) \overline{\otimes} \mathcal{V})(P \otimes I_{H_1}) = P \otimes \mathcal{V}$$

where $P \otimes \mathcal{V} = \{P \otimes A : A \in \mathcal{V}\}$. Now $(P \otimes I_{H_1})(\mathcal{B}(K) \otimes \mathcal{V})(P \otimes I_{H_1}) \subseteq P \otimes \mathcal{V}$ is clear, where $\mathcal{B}(K) \otimes \mathcal{V}$ is the algebraic tensor product, and taking weak* limits therefore establishes the inclusion \subseteq . The reverse inclusion is trivial. This proves the claim and shows that $\mathcal{V} \neq \mathcal{W}$ (hence $P \otimes \mathcal{V} \neq P \otimes \mathcal{W}$) implies $\mathcal{B}(K) \overline{\otimes} \mathcal{V} \neq \mathcal{B}(K) \overline{\otimes} \mathcal{W}$. Moreover, $\mathcal{V} \not\subseteq \mathcal{W}$ implies $\mathcal{B}(K) \overline{\otimes} \mathcal{V} \not\subseteq \mathcal{B}(K) \overline{\otimes} \mathcal{W}$.

Finally, let $\tilde{\mathcal{V}} \subseteq \mathcal{B}(K \otimes H_1)$ be a W^* -bimodule over $\mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$ and let

$$\mathcal{V} = \{A \in \mathcal{B}(H_1) : P \otimes A \in \tilde{\mathcal{V}}\}.$$

To prove (3) we will show that \mathcal{V} is a W^* -bimodule over \mathcal{M}'_1 and $\tilde{\mathcal{V}} = \mathcal{B}(K) \overline{\otimes} \mathcal{V}$. The first part is easy: \mathcal{V} is clearly a weak* closed operator space in $\mathcal{B}(H_1)$, and it is a bimodule over \mathcal{M}'_1 because $\tilde{\mathcal{V}}$ is a bimodule over $P \otimes \mathcal{M}'_1 \subseteq \mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$. For the second part, observe first that $P \otimes \mathcal{V} \subseteq \tilde{\mathcal{V}}$; since $\tilde{\mathcal{V}}$ is a bimodule over $\mathcal{B}(K) \otimes I_{H_1} \subseteq \mathcal{B}(K) \overline{\otimes} \mathcal{M}'_1$, multiplying on the left and the right by operators of the form $B \otimes I_{H_1}$ and taking linear combinations yields $A \otimes \mathcal{V} \subseteq \tilde{\mathcal{V}}$ for any finite rank operator $A \in \mathcal{B}(K)$, and taking weak* limits then shows that $\mathcal{B}(K) \overline{\otimes} \mathcal{V} \subseteq \tilde{\mathcal{V}}$. Conversely, given $A \in \tilde{\mathcal{V}}$ it will suffice to show that

$$(Q \otimes I_{H_1})A(Q \otimes I_{H_1}) \in \mathcal{B}(K) \overline{\otimes} \mathcal{V}$$

for any finite rank projection $Q \in \mathcal{B}(K)$, as these operators converge weak* to A . Then by linearity it is enough to show that

$$(Q_1 \otimes I_{H_1})A(Q_2 \otimes I_{H_1}) \in \mathcal{B}(K) \overline{\otimes} \mathcal{V}$$

for any rank 1 projections Q_1 and Q_2 . But letting V_1 and V_2 be rank 1 partial isometries in $\mathcal{B}(K)$ such that $V_1 V_1^* = V_2 V_2^* = P$, $V_1^* V_1 = Q_1$, and $V_2^* V_2 = Q_2$, we have

$$(V_1 \otimes I_{H_1})A(V_2^* \otimes I_{H_1}) = P \otimes B$$

for some $B \in \mathcal{V}$, and then

$$V_1^* V_2 \otimes B = (V_1^* \otimes I_{H_1})(P \otimes B)(V_2 \otimes I_{H_1}) = (Q_1 \otimes I_{H_1})A(Q_2 \otimes I_{H_1}).$$

So $(Q_1 \otimes I_{H_1})A(Q_2 \otimes I_{H_1})$ does belong to $\mathcal{B}(K) \overline{\otimes} \mathcal{V}$. \square

The following separation lemma will also be useful in the sequel.

Lemma 2.8. *Let \mathcal{V} be a quantum relation on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ and let $A \in \mathcal{B}(H) - \mathcal{V}$. Then there is a pair of projections P and Q in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2) \subseteq \mathcal{B}(H \otimes l^2)$ such that*

$$P(A \otimes I)Q \neq 0$$

but

$$P(B \otimes I)Q = 0$$

for all $B \in \mathcal{V}$.

Proof. Since \mathcal{V} is weak* closed there is a weak* continuous linear functional on $\mathcal{B}(H)$ that annihilates \mathcal{V} but not A . Thus ([14], p. 67) there exist a pair of vectors v and w in $H \otimes l^2$ such that

$$\langle (A \otimes I)w, v \rangle \neq 0$$

but

$$\langle (B \otimes I)w, v \rangle = 0$$

for all $B \in \mathcal{V}$. Moreover, since \mathcal{V} is a bimodule over \mathcal{M}' , we have

$$\langle (B \otimes I)w', v' \rangle = 0$$

for all $B \in \mathcal{V}$, $v' \in (\mathcal{M}' \otimes I)v$, and $w' \in (\mathcal{M}' \otimes I)w$. Let $P, Q \in \mathcal{B}(H \otimes l^2)$ be the orthogonal projections onto the closures of $(\mathcal{M}' \otimes I)v$ and $(\mathcal{M}' \otimes I)w$, respectively. Then we immediately have

$$P(A \otimes I)Q \neq 0$$

and

$$P(B \otimes I)Q = 0$$

for all $B \in \mathcal{V}$. Also, by their construction the ranges of P and Q are invariant for every operator in $\mathcal{M}' \otimes I$, hence P and Q commute with every operator in $\mathcal{M}' \otimes I$, hence $P, Q \in \mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$ ([14], Theorem IV.5.9). \square

2.4. The abelian case. Next we connect quantum relations to measurable relations when \mathcal{M} is abelian. This section is closely related to Arveson's celebrated paper [1], and we will show in the next section that some of Arveson's main results can easily be derived from ours. Actually, the converse is also true, by a simple application of the linking algebra construction: if \mathcal{V} is a quantum relation on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ then

$$\mathcal{A} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in \mathcal{B}(H \oplus H) : A, C \in \mathcal{M}' \text{ and } B \in \mathcal{V} \right\}$$

is a unital weak* closed operator algebra that contains $\mathcal{M}' \oplus \mathcal{M}'$. In this way quantum relations can be converted into operator algebras, and using this device we could without too much effort deduce the main results of this section from [1]. However, we prefer to give direct proofs (which are also not hard, given the machinery we have already built up).

Theorem 2.9. *Let (X, μ) be a finitely decomposable measure space and let $\mathcal{M} \cong L^\infty(X, \mu)$ be the von Neumann algebra of bounded multiplication operators on $L^2(X, \mu)$. If \mathcal{R} is a measurable relation on X then*

$$\mathcal{V}_{\mathcal{R}} = \{A \in \mathcal{B}(L^2(X, \mu)) : (p, q) \notin \mathcal{R} \Rightarrow M_p A M_q = 0\}$$

is a quantum relation on \mathcal{M} ; conversely, if \mathcal{V} is a quantum relation on \mathcal{M} then

$$\mathcal{R}_{\mathcal{V}} = \{(p, q) : M_p A M_q \neq 0 \text{ for some } A \in \mathcal{V}\}$$

is a measurable relation on X . We have $\mathcal{R} = \mathcal{R}_{\mathcal{V}_{\mathcal{R}}}$ for any measurable relation \mathcal{R} on X and $\mathcal{V} \subseteq \mathcal{V}_{\mathcal{R}_{\mathcal{V}}}$ for any quantum relation \mathcal{V} on \mathcal{M} .

If $\mathcal{V}_1, \mathcal{V}_2$, and \mathcal{V}_3 are quantum relations on \mathcal{M} and $\mathcal{R}_i = \mathcal{R}_{\mathcal{V}_i}$ ($i = 1, 2, 3$) then

- (a) $\mathcal{V}_1 \subseteq \mathcal{V}_2 \Rightarrow \mathcal{R}_1 \subseteq \mathcal{R}_2$
- (b) \mathcal{V}_1 is the diagonal quantum relation $\Leftrightarrow \mathcal{R}_1$ is the diagonal measurable relation
- (c) \mathcal{V}_1 is the transpose of $\mathcal{V}_2 \Rightarrow \mathcal{R}_1$ is the transpose of \mathcal{R}_2
- (d) \mathcal{V}_3 is the product of \mathcal{V}_1 and $\mathcal{V}_2 \Rightarrow \mathcal{R}_3$ is the product of \mathcal{R}_1 and \mathcal{R}_2 .

Proof. Note first that $\mathcal{M} = \mathcal{M}'$ in this case. Let \mathcal{R} be a measurable relation on X . It is clear that $\mathcal{V}_{\mathcal{R}}$ is a linear subspace of $\mathcal{B}(L^2(X, \mu))$. Also, $\mathcal{V}_{\mathcal{R}}$ is weak operator closed and therefore weak* closed. Finally, if M_f and M_g are any two multiplication operators in \mathcal{M} then

$$M_p A M_q = 0 \Rightarrow M_p (M_f A M_g) M_q = M_f (M_p A M_q) M_g = 0,$$

which shows that $A \in \mathcal{V}_{\mathcal{R}} \Rightarrow M_f A M_g \in \mathcal{V}_{\mathcal{R}}$. So $\mathcal{V}_{\mathcal{R}}$ is a W*-bimodule over $\mathcal{M}' = \mathcal{M}$.

Next, let \mathcal{V} be a quantum relation on \mathcal{M} . We verify that $\mathcal{R}_{\mathcal{V}}$ satisfies the pair of conditions stated in Definition 1.2. First, if $p' \leq p$, $q' \leq q$, and $M_{p'} A M_{q'} \neq 0$ then it is clear that $M_p A M_q \neq 0$. Second, say $p = \bigvee p_\lambda$ and $q = \bigvee q_\kappa$ and suppose $M_{p_\lambda} A M_{q_\kappa} = 0$ for all λ and κ . Then $\langle A w, v \rangle = 0$ for all v in the range of any M_{p_λ} and all w in the range of any M_{q_κ} . Taking linear combinations and norm limits then yields $\langle A w, v \rangle = 0$ for all v in the range of M_p and all w in the range of M_q , so that $M_p A M_q = 0$. This verifies the second condition and shows that $\mathcal{R}_{\mathcal{V}}$ is a measurable relation.

Now let \mathcal{R} be a measurable relation, let $\mathcal{V} = \mathcal{V}_{\mathcal{R}}$, and let $\tilde{\mathcal{R}} = \mathcal{R}_{\mathcal{V}}$. It is immediate that $\tilde{\mathcal{R}} \subseteq \mathcal{R}$. For the reverse inclusion, let $(p, q) \in \mathcal{R}$ and say $p = \chi_S$ and $q = \chi_T$. By Theorem 1.13 there exists a nonzero bounded operator $A : L^2(T, \mu|_T) \rightarrow L^2(S, \mu|_S)$ such that $(p', q') \notin \mathcal{R}$ implies $M_{p'} A M_{q'} = 0$. Extending A

to be zero on $L^2(X - T, \mu|_{X-T})$, we get an operator $\tilde{A} \in B(L^2(X, \mu))$ which satisfies $M_p \tilde{A} M_q \neq 0$ and $M_{p'} \tilde{A} M_{q'} = 0$ for all projections p' and q' such that $(p', q') \notin \mathcal{R}$. Then $\tilde{A} \in \mathcal{V}$ and this shows that $(p, q) \in \tilde{\mathcal{R}}$, so we conclude that $\mathcal{R} = \tilde{\mathcal{R}}$.

All of the remaining assertions except for part (d) and the reverse implication in (b) are straightforward. For the reverse implication in (b), observe that $\mathcal{R}_1 = \Delta$ implies that $(1-p, p) \notin \mathcal{R}_1$ for any p , and hence that every operator in \mathcal{V}_1 commutes with every projection in \mathcal{M} . Since \mathcal{M} is maximal abelian, it follows that \mathcal{V}_1 is contained in, and therefore a weak* closed ideal of, \mathcal{M} . Thus $\mathcal{V}_1 = P\mathcal{M}$ for some projection P in \mathcal{M} , and if P is not the identity then it is easy to see that we could not have $\mathcal{R}_1 = \Delta$. Thus $\mathcal{V}_1 = \mathcal{M}$ is the diagonal quantum relation.

For part (d), suppose $\mathcal{V}_3 = \mathcal{V}_1 \mathcal{V}_2$ and let $(p, r) \in \mathcal{R}_3$. Then there exist $A \in \mathcal{V}_1$ and $B \in \mathcal{V}_2$ such that $M_p A B M_r \neq 0$. For any projection q in $L^\infty(X, \mu)$, we therefore have either $M_p A M_q \neq 0$, or else

$$M_p A M_q = 0 \quad \Rightarrow \quad M_p A = M_p A M_{1-q} \quad \Rightarrow \quad M_p A B M_r = M_p A M_{1-q} B M_r.$$

Since $M_p A B M_r \neq 0$, the latter implies $M_{1-q} B M_r \neq 0$. Thus we have shown that either $(p, q) \in \mathcal{R}_1$ or $(1-q, r) \in \mathcal{R}_2$, and we conclude that $(p, r) \in \mathcal{R}_1 \mathcal{R}_2$. So $\mathcal{R}_3 \subseteq \mathcal{R}_1 \mathcal{R}_2$. Conversely, suppose $(p, r) \in \mathcal{R}_1 \mathcal{R}_2$. Then there is a nonzero projection q in $L^\infty(X, \mu)$ such that $(p, q') \in \mathcal{R}_1$ and $(q', r) \in \mathcal{R}_2$ for every nonzero $q' \leq q$. Now observe that the set of vectors $w \in L^2(X, \mu)$ such that $M_p A w = 0$ for all $A \in \mathcal{V}_1$ is a closed subspace that is invariant for \mathcal{M} (since \mathcal{V}_1 is a right \mathcal{M} -module); therefore it is the range of a projection r in $\mathcal{M}' = \mathcal{M}$. Then $M_p A M_r = 0$ for all $A \in \mathcal{V}_1$, so $(p, r) \notin \mathcal{R}_1$ and hence $r \leq 1-q$. This shows that for every nonzero w in the range of M_q we have $M_p A w \neq 0$ for some $A \in \mathcal{V}_1$. Now $(q, r) \in \mathcal{R}_2$ implies that $M_q B M_r \neq 0$ for some $B \in \mathcal{V}_2$, i.e., $M_q B M_r v \neq 0$ for some vector v . The preceding comment then shows that $M_p A M_q B M_r \neq 0$ for some $A \in \mathcal{V}_1$ and $B \in \mathcal{V}_2$, so that $(p, r) \in \mathcal{R}_3$ (since $A M_q B \in \mathcal{V}_1 \mathcal{V}_2 = \mathcal{V}_3$). Thus $\mathcal{R}_3 = \mathcal{R}_1 \mathcal{R}_2$. \square

In general we do not have $\mathcal{V} = \mathcal{V}_{\mathcal{R}_\mathcal{V}}$. We will see below (Corollary 2.16 and Proposition 2.19) that if $\mathcal{V} = \mathcal{A}$ is an operator algebra that contains a maximal abelian von Neumann algebra then this happens precisely if \mathcal{A} is reflexive in the sense that $\mathcal{A} = \text{Alg}(\text{Lat}(\mathcal{A}))$, and Arveson ([1], Section 2.5; see also [6]) has given an example of a weak operator closed operator algebra that contains (and hence is a bimodule over) a maximal abelian von Neumann algebra but is not reflexive. In general, we will see that $\mathcal{V} = \mathcal{V}_{\mathcal{R}_\mathcal{V}}$ if and only if \mathcal{V} is reflexive in the sense of Loginov and Sul'man ([3], Section 15.B); see Corollary 2.16.

In any case, by Theorem 2.9 $\mathcal{V}_\mathcal{R}$ is the maximal quantum relation on \mathcal{M} associated to the measurable relation \mathcal{R} . There is also always a minimal quantum relation associated to \mathcal{R} . To describe it, recall first ([14], p. 257) that the map $f \otimes v \mapsto f \cdot v$ implements an isometric isomorphism of the Hilbert space tensor product $L^2(X, \mu) \otimes l^2$ with the space $L^2(X; l^2)$ of weakly measurable functions $h : X \rightarrow l^2$, up to modification on null sets, such that $\int \|h(x)\|^2 d\mu$ is finite. ("Weakly measurable" means that for every $v \in H$ the scalar-valued function $x \mapsto \langle h(x), v \rangle$ is measurable. Note that this implies that the function

$$x \mapsto \|h(x)\|^2 = \sum_n |\langle h(x), e_n \rangle|^2$$

is measurable, where $\{e_n\}$ is the standard basis of l^2 .) In the following we will identify $L^2(X, \mu) \otimes l^2$ with $L^2(X; l^2)$.

Now given a measurable relation \mathcal{R} on X and $h, k \in L^2(X; l^2)$, say that h is \mathcal{R} -orthogonal to k , and write $h \perp_{\mathcal{R}} k$, if

$$\inf \{ |\langle k(y), h(x) \rangle| : x \in S, y \in T \} = 0$$

for any $S, T \subseteq X$ such that $(\chi_S, \chi_T) \in \mathcal{R}$. It should be understood that this means the infimum must be zero irrespective of any modification of h and k on null sets. Equivalently, $h \not\perp_{\mathcal{R}} k$ if there exists $\epsilon > 0$ and $S, T \subseteq X$ such that $(\chi_S, \chi_T) \in \mathcal{R}$ and $|\langle k(y), h(x) \rangle| \geq \epsilon$ for $x \in S$ and $y \in T$.

Lemma 2.10. *Let $S, T \subseteq X$, $h \in L^2(S; l^2)$, $k \in L^2(T; l^2)$, and $\epsilon > 0$, and suppose $\mu(S), \mu(T) < \infty$ and $h \perp_{\mathcal{R}} k$. Then there exist partitions $\{S_1, \dots, S_m\}$ and $\{T_1, \dots, T_n\}$ of S and T and simple functions*

$$h_\epsilon = \sum_{i=1}^m \chi_{S_i} \cdot v_i \quad \text{and} \quad k_\epsilon = \sum_{j=1}^n \chi_{T_j} \cdot w_j$$

($v_i, w_j \in l^2$) such that (1) $\|h - h_\epsilon\|, \|k - k_\epsilon\| \leq \epsilon$ and (2) $|\langle w_j, v_i \rangle| \leq \epsilon$ for any i and j such that $(\chi_{S_i}, \chi_{T_j}) \in \mathcal{R}$.

Proof. First find N large enough that

$$h_N = \chi_{\{x: \|h(x)\| \leq N\}} \cdot h \quad \text{and} \quad k_N = \chi_{\{y: \|k(y)\| \leq N\}} \cdot k$$

satisfy $\|h - h_N\|, \|k - k_N\| \leq \epsilon/3$, and note that h_N and k_N are still \mathcal{R} -orthogonal. Then since l^2 is separable we can uniformly approximate h_N and k_N with functions of the form

$$h'_\epsilon = \sum_{i=1}^\infty \chi_{S_i} \cdot v_i \quad \text{and} \quad k'_\epsilon = \sum_{j=1}^\infty \chi_{T_j} \cdot w_j$$

where the S_i partition S and the T_j partition T . If the uniform approximation is sufficiently close then (since h_N and k_N are bounded) \mathcal{R} -orthogonality of h_N and k_N will imply that $|\langle w_j, v_i \rangle| \leq \epsilon$ whenever $(\chi_{S_i}, \chi_{T_j}) \in \mathcal{R}$, and we will also have $\|h_N - h'_\epsilon\|, \|k_N - k'_\epsilon\| < \epsilon/3$. Finally, define h_ϵ and k_ϵ by truncating the sums that define h'_ϵ and k'_ϵ ; that is, for suitable m and n replace S_m and T_n with $\bigcup_{i \geq m} S_i$ and $\bigcup_{j \geq n} T_j$ and take $v_m = w_n = 0$. This can be done at a cost in norm of at most $\epsilon/3$, so we will have achieved conditions (1) and (2). \square

Theorem 2.11. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space (X, μ) and let $\mathcal{M} \cong L^\infty(X, \mu)$ be the von Neumann algebra of bounded multiplication operators on $L^2(X, \mu)$. Then*

$$\tilde{\mathcal{V}}_{\mathcal{R}} = \{A \in \mathcal{B}(L^2(X, \mu)) : h \perp_{\mathcal{R}} k \Rightarrow \langle (A \otimes I)k, h \rangle = 0\},$$

with h and k ranging over $L^2(X, \mu) \otimes l^2 \cong L^2(X; l^2)$, is a quantum relation on \mathcal{M} whose associated measurable relation (Theorem 2.9) is \mathcal{R} . If \mathcal{V} is any quantum relation on \mathcal{M} whose associated measurable relation contains \mathcal{R} then $\tilde{\mathcal{V}}_{\mathcal{R}} \subseteq \mathcal{V}$.

Proof. It is easy to see that $\tilde{\mathcal{V}}_{\mathcal{R}}$ is a weak* closed linear subspace of $\mathcal{B}(L^2(X, \mu))$. To check that it is a bimodule over \mathcal{M} , let $A \in \tilde{\mathcal{V}}_{\mathcal{R}}$ and $f, g \in L^\infty(X, \mu)$ and suppose $h \perp_{\mathcal{R}} k$; then $\bar{f} \cdot h \perp_{\mathcal{R}} g \cdot k$ and so

$$\langle (M_f A M_g \otimes I)k, h \rangle = \langle (A \otimes I)(g \cdot k), \bar{f} \cdot h \rangle = 0,$$

showing that $M_f A M_g \in \tilde{\mathcal{V}}_{\mathcal{R}}$. So $\tilde{\mathcal{V}}_{\mathcal{R}}$ is a quantum relation on \mathcal{M} .

Next, we show that the measurable relation associated to $\tilde{\mathcal{V}}_{\mathcal{R}}$ is \mathcal{R} . Let $S, T \subseteq X$ and suppose $(\chi_S, \chi_T) \notin \mathcal{R}$. Then for any $f \in L^2(S, \mu|_S)$ and $g \in L^2(T, \mu|_T)$ and any unit vector $v \in l^2$ we have $f \cdot v \perp_{\mathcal{R}} g \cdot v$, so if $A \in \tilde{\mathcal{V}}_{\mathcal{R}}$ then

$$\langle Ag, f \rangle = \langle (A \otimes I)(g \cdot v), f \cdot v \rangle = 0.$$

It follows that $M_{\chi_S} A M_{\chi_T} = 0$ and this shows that the measurable relation associated to $\tilde{\mathcal{V}}_{\mathcal{R}}$ is contained in \mathcal{R} . Conversely, suppose $(\chi_S, \chi_T) \in \mathcal{R}$; we must find $A \in \tilde{\mathcal{V}}_{\mathcal{R}}$ such that $M_{\chi_S} A M_{\chi_T} \neq 0$. Find finite measure subsets $S' \subseteq S$ and $T' \subseteq T$ such that $(\chi_{S'}, \chi_{T'}) \in \mathcal{R}$; it will suffice to show that the operator A with $M_{\chi_{S'}} A M_{\chi_{T'}} \neq 0$ provided by Theorem 1.13 belongs to $\tilde{\mathcal{V}}_{\mathcal{R}}$. Fix $h, k \in L^2(X; l^2)$ such that $h \perp_{\mathcal{R}} k$; we want $\langle (A \otimes I)k, h \rangle = 0$. We show this by considering the approximating operators $A_{\mathcal{S}, \mathcal{T}}$ defined for finite partitions of S' and T' .

Let $h' = \chi_{S'} \cdot h$ and $k' = \chi_{T'} \cdot k$ and let $\epsilon > 0$. Then apply Lemma 2.10 to S', T', h', k' . Now if $A_{\mathcal{S}, \mathcal{T}}$ is the operator constructed in the proof of Theorem 1.13 for any finite partitions \mathcal{S} and \mathcal{T} of S' and T' which are subordinate to $\{S_i\}$ and $\{T_j\}$ then

$$|\langle (A_{\mathcal{S}, \mathcal{T}} \otimes I)k_{\epsilon}, h_{\epsilon} \rangle| = \left| \sum \langle w'_i, v'_i \rangle \mu(S'_i) \right| \leq \epsilon \mu(S').$$

(Here $\{S'_1, \dots, S'_{m'}\}$ and $\{T'_1, \dots, T'_{n'}\}$ are the refined partitions produced in the construction of $A_{\mathcal{S}, \mathcal{T}}$ and $h_{\epsilon} = \sum_{i=1}^{m'} \chi_{S'_i} \cdot v'_i$ and $k_{\epsilon} = \sum_{j=1}^{n'} \chi_{T'_j} \cdot w'_j$ are the corresponding expressions for h_{ϵ} and k_{ϵ} .) Taking the limit in \mathcal{S} and \mathcal{T} then yields $|\langle (A \otimes I)k_{\epsilon}, h_{\epsilon} \rangle| \leq \epsilon \mu(S')$ and taking $\epsilon \rightarrow 0$ yields $\langle (A \otimes I)k, h \rangle = 0$, as desired.

Now let \mathcal{V} be any quantum relation on \mathcal{M} whose associated measurable relation contains \mathcal{R} . If $\tilde{\mathcal{V}}_{\mathcal{R}} \subsetneq \mathcal{V}$ then as in the proof of Lemma 2.8 there must exist $h, k \in L^2(X; l^2)$ such that $\langle (A \otimes I)k, h \rangle \neq 0$ for some $A \in \tilde{\mathcal{V}}_{\mathcal{R}}$ — and hence $h \not\perp_{\mathcal{R}} k$ — but $\langle (B \otimes I)k, h \rangle = 0$ for all $B \in \mathcal{V}$. Thus suppose $h, k \in L^2(X; l^2)$ are not \mathcal{R} -orthogonal; we complete the proof by showing that there exists $B \in \mathcal{V}$ such that $\langle (B \otimes I)k, h \rangle \neq 0$. Our argument is a straightforward adaptation of the ingenious proof of Theorem 2.1.5 in [1]. First, since $h \not\perp_{\mathcal{R}} k$ there exist $S, T \subseteq X$ and $\epsilon > 0$ such that $(\chi_S, \chi_T) \in \mathcal{R}$ and $|\langle k(y), h(x) \rangle| \geq \epsilon$ for all $x \in S$ and $y \in T$. For some N we must have $(\chi_S, \chi_{T_N}) \in \mathcal{R}$ where $T_N = \{y \in T : \|k(y)\| \leq N\}$, so we can assume that k is bounded on T . By scaling k (which could change the value of ϵ), we may suppose $\|k(y)\| \leq 1$ for all $y \in T$. Now find a countable partition $\{S_i\}$ of S together with a sequence $\{v_i\} \subseteq l^2$ such that $v_i \in h(S_i) \subseteq \text{ball}(v_i, \epsilon/2)$ for all i . Then we must have $(\chi_{S_i}, \chi_T) \in \mathcal{R}$ for some i and j . Without loss of generality we may then replace S and h with S_i and $\chi_{S_i} \cdot h$. In particular, we can assume that there is a vector $v = v_i \in l^2$ such that $\|h(x) - v\| \leq \epsilon/2$ for all $x \in S$ and $|\langle k(y), v \rangle| \geq \epsilon$ for all $y \in T$.

Let $P, Q \in \mathcal{M} \overline{\otimes} \mathcal{B}(l^2) \cong L^\infty(X; \mathcal{B}(l^2))$ ([14], Theorem IV.7.17) respectively be the orthogonal projections onto $(\mathcal{M} \otimes I)h$ and $(\mathcal{M} \otimes I)k$. Suppose for the sake of contradiction that $P(A \otimes I)Q = 0$ for all $A \in \mathcal{V}$. Then for any $w \in L^2(X, \mu)$ we have

$$\|P(A \otimes I)(w \otimes v)\|^2 = \|P(A \otimes I)(I - Q)(w \otimes v)\|^2 \leq \|A\|^2 \|(I - Q)(w \otimes v)\|^2;$$

letting $f(x) = \|P(\chi_X \cdot v)(x)\|^2$ and $g(x) = \|(I - Q)(\chi_X \cdot v)(x)\|^2$ (both in $L^\infty(X, \mu)$), this can be expressed as $A^* M_f A \leq \|A\|^2 M_g$, since

$$\langle A^* M_f A w, w \rangle = \int \|P(\chi_X \cdot v)(x)\|^2 \|(Aw)(x)\|^2 d\mu$$

$$\begin{aligned}
&= \int \|P(Aw \otimes v)(x)\|^2 d\mu \\
&= \|P(A \otimes I)(w \otimes v)\|^2
\end{aligned}$$

and similarly $\langle M_g w, w \rangle = \|(I - Q)(w \otimes v)\|^2$. This inequality holds for all $A \in \mathcal{V}$.

Now given $A \in \mathcal{V}$ and $\delta > 0$, let $B = M_{\sqrt{\mathcal{I}}} A M_{1/\sqrt{g+\delta}} \in \mathcal{V}$. We then have

$$B^* B = M_{1/\sqrt{g+\delta}} A^* M_f A M_{1/\sqrt{g+\delta}} \leq \|A\|^2 M_{1/\sqrt{g+\delta}} M_g M_{1/\sqrt{g+\delta}} = \|A\|^2 M_{g/(g+\delta)}$$

so that $\|B\| \leq \|A\|$, and hence

$$(M_{1/\sqrt{g+\delta}} A^* M_{\sqrt{\mathcal{I}}}) M_f (M_{\sqrt{\mathcal{I}}} A M_{1/\sqrt{g+\delta}}) = B^* M_f B \leq \|B\|^2 M_g \leq \|A\|^2 M_g.$$

Multiplying on both sides by $M_{\sqrt{g+\delta}}$ then yields $A^* M_{f^2} A \leq \|A\|^2 M_{g(g+\delta)}$, and taking $\delta \rightarrow 0$, we get $A^* M_{f^2} A \leq \|A\|^2 M_{g^2}$. Applying this argument inductively establishes that $A^* M_{f^n} A \leq \|A\|^2 M_{g^n}$ for all $A \in \mathcal{V}$ and all $n \in \mathbf{N}$. Now $h \in \text{ran}(P)$ implies that $f(x) \geq \|v\|^2 - (\epsilon/2)^2$ for all $x \in S$, while $k \in \text{ran}(Q)$ implies that $g(y) \leq \|v\|^2 - \epsilon^2$ for all $y \in T$. Thus

$$\begin{aligned}
(\|v\|^2 - (\epsilon/2)^2)^n A^* M_{\chi_S} A &\leq A^* M_{f^n} A \leq \|A\|^2 M_{g^n} \\
&\leq \|A\|^2 [(\|v\|^2 - \epsilon^2)^n M_{\chi_T} + \|v\|^2 M_{\chi_{X-T}}]
\end{aligned}$$

and so

$$M_{\chi_T} A^* M_{\chi_S} A M_{\chi_T} \leq \|A\|^2 \left(\frac{\|v\|^2 - \epsilon^2}{\|v\|^2 - (\epsilon/2)^2} \right)^n M_{\chi_T}.$$

Taking $n \rightarrow \infty$ then yields $M_{\chi_T} A^* M_{\chi_S} A M_{\chi_T} = 0$, and hence $M_{\chi_S} A M_{\chi_T} = 0$. Since this is true for all $A \in \mathcal{V}$ we cannot have $(\chi_S, \chi_T) \in \mathcal{R}$, a contradiction. We conclude that $P(A \otimes I)Q \neq 0$ for some $A \in \mathcal{V}$, and hence that $\langle (B \otimes I)k, h \rangle \neq 0$ for some $B \in \mathcal{V}$. This completes the proof. \square

2.5. Operator reflexivity. Specializing the preceding work to the case of measurable partial orders, we recover Arveson's basic results on commutative subspace lattices. In particular, the formula $\mathcal{R} = \mathcal{R}_{\mathcal{V}_{\mathcal{R}}}$ in Theorem 2.9 emerges as an attractive generalization of Arveson's reflexivity theorem (from which it can, alternatively, be deduced; see the comment at the beginning of Section 2.4).

Theorem 2.12. ([1], Theorem 1.3.1) *Let \mathcal{L} be a complete 0,1-lattice of commuting projections in some $\mathcal{B}(H)$. Then there is a measurable preorder \mathcal{R} on a finitely decomposable measure space (X, μ) and an isomorphism $H \cong L^2(X, \mu)$ that takes \mathcal{L} to*

$$\{M_{\chi_S} : S \text{ is a lower set for } \mathcal{R}\}.$$

Proof. \mathcal{L} generates an abelian von Neumann algebra and hence is contained in a maximal abelian von Neumann algebra \mathcal{M} . Then there exists a finitely decomposable measure space (X, μ) and an isomorphism $H \cong L^2(X, \mu)$ that takes \mathcal{M} to the algebra of bounded multiplication operators. That is, $\mathcal{M} \cong L^\infty(X, \mu)$. This isomorphism takes \mathcal{L} to a complete 0,1-sublattice of the lattice of projections in $L^\infty(X, \mu)$ and the result now follows from Theorem 1.9. \square

(Theorem 1.3.1 of [1] is expressed in terms of pointwise preorders; this version, when μ is σ -finite, follows from Theorem 1.10 and the comment preceding that result.)

For any set of projections $\mathcal{L} \subset \mathcal{B}(H)$ let $\text{Alg}(\mathcal{L})$ be the algebra of operators for which the range of every projection in \mathcal{L} is invariant; that is, $\text{Alg}(\mathcal{L}) = \{A \in \mathcal{B}(H) :$

$PAP = AP$ for all $P \in \mathcal{L}$. For any set of operators $\mathcal{A} \subseteq \mathcal{B}(H)$ let $\text{Lat}(\mathcal{A})$ be the lattice of projections whose range is invariant for every operator in \mathcal{A} .

Theorem 2.13. ([1], Theorem 1.6.1) *Let \mathcal{L} be a complete 0,1-lattice of commuting projections in some $\mathcal{B}(H)$. Then $\mathcal{L} = \text{Lat}(\text{Alg}(\mathcal{L}))$.*

Proof. By Theorem 2.12 we may assume that $H = L^2(X, \mu)$ and there is a measurable preorder \mathcal{R} on X such that \mathcal{L} consists of the operators M_p for p a projection in $L^\infty(X, \mu)$ satisfying $(1 - p, p) \notin \mathcal{R}$. Define $\mathcal{V}_{\mathcal{R}}$ as in Theorem 2.9; we claim that $\mathcal{V}_{\mathcal{R}} = \text{Alg}(\mathcal{L})$. To see this first let $A \in \mathcal{V}_{\mathcal{R}}$ and $M_p \in \mathcal{L}$. Then $(1 - p, p) \notin \mathcal{R}$, so $M_{1-p}AM_p = 0$, which shows that $A \in \text{Alg}(\mathcal{L})$. Conversely, let $A \in \text{Alg}(\mathcal{L})$. If $p, q \in L^\infty(X, \mu)$ satisfy $(p, q) \notin \mathcal{R}$ then there exists $M_{q'} \in \mathcal{L}$ such that $q \leq q'$ and $pq' = 0$ (Theorem 1.9), and $M_{q'}AM_{q'} = AM_{q'}$ then implies that

$$M_pAM_q = M_pAM_{q'}M_q = M_pM_{q'}AM_{q'}M_q = 0,$$

which shows that $A \in \mathcal{V}_{\mathcal{R}}$. This proves the claim.

Observe that $\text{Alg}(\mathcal{L})$ contains all bounded multiplication operators, so any projection in $\mathcal{B}(H)$ whose range is invariant for $\text{Alg}(\mathcal{L})$ must commute with all multiplication operators and hence must have the form M_p with $p \in L^\infty(X, \mu)$. Now by Theorem 2.9 we have $\mathcal{R} = \{(p, q) : M_pAM_q \neq 0 \text{ for some } A \in \mathcal{V}_{\mathcal{R}}\}$. So

$$\begin{aligned} M_p \in \mathcal{L} &\Leftrightarrow (1 - p, p) \notin \mathcal{R} \\ &\Leftrightarrow M_{1-p}AM_p = 0 \text{ for all } A \in \mathcal{V}_{\mathcal{R}} \\ &\Leftrightarrow M_pAM_p = AM_p \text{ for all } A \in \text{Alg}(\mathcal{L}). \end{aligned}$$

This shows that a projection of the form M_p belongs to \mathcal{L} if and only if the range of M_p is invariant for $\text{Alg}(\mathcal{L})$. We saw just above that every projection in $\text{Lat}(\text{Alg}(\mathcal{L}))$ must take this form, so we conclude that $\mathcal{L} = \text{Lat}(\text{Alg}(\mathcal{L}))$. \square

Next we relate our approach to Loginov and Sul'man's generalized notion of reflexivity ([3], Section 15.B). We know from Lemma 2.8 that a quantum relation \mathcal{V} is determined by the pairs of projections P and Q in $\mathcal{B}(H \otimes l^2)$ that annihilate it (i.e., such that $P(A \otimes I)Q = 0$ for all $A \in \mathcal{V}$). We also noted in the comment following Theorem 2.9 that \mathcal{V} in general is not determined by the pairs of projections in $\mathcal{B}(H)$ that annihilate it. This suggests the following definition:

Definition 2.14. A subspace $\mathcal{V} \subseteq \mathcal{B}(H)$ is *operator reflexive* if

$$\mathcal{V} = \{B \in \mathcal{B}(H) : PVQ = 0 \Rightarrow PBQ = 0\},$$

with P and Q ranging over projections in $\mathcal{B}(H)$.

We use the term “operator reflexive” to avoid confusion with the notion of reflexivity of a quantum relation (Definition 2.4 (d)).

Definition 2.14 makes sense for any subspace \mathcal{V} , but in the case of quantum relations it can be slightly modified:

Proposition 2.15. *Let \mathcal{V} be a quantum relation over a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. Then \mathcal{V} is operator reflexive if and only if*

$$\mathcal{V} = \{B \in \mathcal{B}(H) : PVQ = 0 \Rightarrow PBQ = 0\},$$

with P and Q ranging over projections in \mathcal{M} .

Proof. Let P and Q be projections in $\mathcal{B}(H)$ and suppose $P\mathcal{V}Q = 0$. Since \mathcal{V} is a bimodule over \mathcal{M}' , we have $P\mathcal{M}'\mathcal{V}\mathcal{M}'Q = 0$, and hence $\tilde{P}\mathcal{V}\tilde{Q} = 0$ where \tilde{P} is the orthogonal projection onto the closure of $\mathcal{M}'(\text{ran}(P))$ and \tilde{Q} is the orthogonal projection onto the closure of $\mathcal{M}'(\text{ran}(Q))$. Also, \tilde{P} and \tilde{Q} belong to \mathcal{M} because their ranges are invariant for \mathcal{M}' . So for any projections P and Q such that $P\mathcal{V}Q = 0$ there are larger projections \tilde{P} and \tilde{Q} in \mathcal{M} such that $\tilde{P}\mathcal{V}\tilde{Q} = 0$. This entails that the two conditions are equivalent. \square

Operator reflexivity is of particular interest for quantum relations over maximal abelian von Neumann algebras because of the following result.

Corollary 2.16. *Let (X, μ) be a finitely decomposable measure space, let $\mathcal{M} \cong L^\infty(X, \mu)$ be the von Neumann algebra of bounded multiplication operators on $L^2(X, \mu)$, and let \mathcal{V} be a quantum relation on \mathcal{M} . Then in the notation of Theorem 2.9, $\mathcal{V} = \mathcal{V}_{\mathcal{R}_\mathcal{V}}$ if and only if \mathcal{V} is operator reflexive.*

The proof of this corollary is trivial, as $\mathcal{V}_{\mathcal{R}_\mathcal{V}}$ by definition consists of precisely those operators B which satisfy $P\mathcal{V}Q = 0 \Rightarrow PBQ = 0$, with P and Q ranging over projections in \mathcal{M} .

Loginov and Sul'man's version of operator reflexivity is stated in part (iii) of the following result. Our definition is also equivalent to one formulated by Erdos [5]. Given a subspace $\mathcal{V} \subseteq \mathcal{B}(H)$, for any projection $Q \in \mathcal{B}(H)$ let $\phi(Q)$ be the orthogonal projection onto the closure of $\mathcal{V}(\text{ran}(Q))$. That is, $\phi(Q) = I - \bigvee\{P : P\mathcal{V}Q = 0\}$. Erdos's definition is stated in part (ii) of the next result. Part (v) is Larson's characterization of operator reflexivity ([11], Lemma 2).

Proposition 2.17. *Let \mathcal{V} be a subspace of $\mathcal{B}(H)$. The following are equivalent:*

- (i) \mathcal{V} is operator reflexive
- (ii) $\mathcal{V} = \{B \in \mathcal{B}(H) : BQ = \phi(Q)BQ \text{ for all projections } Q \in \mathcal{B}(H)\}$
- (iii) $\mathcal{V} = \{B \in \mathcal{B}(H) : Bv \in \overline{\mathcal{V}v} \text{ for all } v \in H\}$
- (iv) for any $A \in \mathcal{B}(H) - \mathcal{V}$ there exist $v, w \in H$ such that

$$\langle Aw, v \rangle \neq 0$$

but

$$\langle Bw, v \rangle = 0$$

for all $B \in \mathcal{V}$

- (v) \mathcal{V} is weak* closed and its preannihilator $\mathcal{V}_\perp \subseteq \mathcal{TC}(H)$ is generated by rank one operators.

Proof. (i) \Leftrightarrow (ii): This follows from the fact that $P\mathcal{V}Q = 0 \Leftrightarrow P \leq I - \phi(Q)$.

(ii) \Leftrightarrow (iii): Trivial.

(i) \Leftrightarrow (iv): $P\mathcal{V}Q = 0 \Rightarrow PBQ = 0$ holds for all projections P and Q if and only if it holds for all rank one projections P and Q , and $\langle Aw, v \rangle = 0 \Leftrightarrow PAQ = 0$ where P and Q are respectively the orthogonal projections onto $\mathbf{C}v$ and $\mathbf{C}w$.

(iv) \Leftrightarrow (v): The linear functionals $A \mapsto \text{tr}(AB)$ on $\mathcal{B}(H)$ with B a rank one operator are precisely the linear functionals $A \mapsto \langle Aw, v \rangle$ with $v, w \in H$. \square

Every subspace \mathcal{V} of $\mathcal{B}(H)$ has a *reflexive closure*

$$\overline{\mathcal{V}} = \{B \in \mathcal{B}(H) : P\mathcal{V}Q = 0 \Rightarrow PBQ = 0\},$$

with P and Q ranging over projections in $\mathcal{B}(H)$. It is easy to see that $\overline{\mathcal{V}}$ is the smallest operator reflexive subspace that contains \mathcal{V} . Moreover, if \mathcal{V} is a quantum relation then so is $\overline{\mathcal{V}}$:

Proposition 2.18. *Let \mathcal{V} be a quantum relation on a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$. Then its reflexive closure $\overline{\mathcal{V}}$ is also a quantum relation on \mathcal{M} , and we have*

$$\overline{\mathcal{V}} = \{B \in \mathcal{B}(H) : P\mathcal{V}Q = 0 \Rightarrow PBQ = 0\}$$

with P and Q ranging over projections in \mathcal{M} .

Proof. The first statement follows from the second because if $B \in \overline{\mathcal{V}}$, $A, C \in \mathcal{M}'$, and $PBQ = 0$ for all projections $P, Q \in \mathcal{M}$ with $P\mathcal{V}Q = 0$ then

$$P(ABC)Q = A(PBQ)C = 0$$

for all such projections P and Q . This shows that $ABC \in \overline{\mathcal{V}}$.

The second assertion of the proposition follows from the observation made in the proof of Proposition 2.15 that if $P\mathcal{V}Q = 0$ for some projections $P, Q \in \mathcal{B}(H)$ then there exist projections $\tilde{P}, \tilde{Q} \in \mathcal{M}$ with $P \leq \tilde{P}$, $Q \leq \tilde{Q}$, and $\tilde{P}\mathcal{V}\tilde{Q} = 0$. \square

Next we note that if \mathcal{V} is an operator algebra then our definition of operator reflexivity is equivalent to the standard one. This follows from Proposition 2.17 (i) \Leftrightarrow (iii) above.

Proposition 2.19. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a unital operator algebra. Then \mathcal{A} is operator reflexive if and only if $\mathcal{A} = \text{Alg}(\text{Lat}(\mathcal{A}))$.*

The next result is given in Lemma 15.4 of [3], but for the sake of completeness we include a short proof here.

Proposition 2.20. ([3], Lemma 15.4) *Let \mathcal{V} be a weak* closed subspace of $\mathcal{B}(H)$. Then $\mathcal{V} \otimes I$ is an operator reflexive subspace of $\mathcal{B}(H \otimes l^2)$.*

Proof. Let $A \in \mathcal{B}(H \otimes l^2) - \mathcal{V} \otimes I$. By Proposition 2.17 (iv) it will suffice to find $v, w \in H \otimes l^2$ such that $\langle Aw, v \rangle \neq 0$ but $\langle (B \otimes I)w, v \rangle = 0$ for all $B \in \mathcal{V}$. There are two cases. First, suppose $A \notin \mathcal{B}(H) \otimes I$. Since $\mathcal{B}(H) \otimes I$ is a von Neumann algebra it is operator reflexive (Proposition 2.19), so by Proposition 2.17 there exist $v, w \in H \otimes l^2$ such that $\langle Aw, v \rangle \neq 0$ but $\langle (B \otimes I)w, v \rangle = 0$ for all $B \in \mathcal{B}(H)$, in particular for all $B \in \mathcal{V}$, as desired. The other case is that $A \in \mathcal{B}(H) \otimes I$, say $A = A_0 \otimes I$. Then $A_0 \notin \mathcal{V}$ and, as in Lemma 2.8, the desired pair of vectors $v, w \in H \otimes l^2$ exist since \mathcal{V} is weak* closed. This completes the proof. \square

We also recover Erdos's generalization of Arveson's theorem on the operator reflexivity of commutative subspace lattices. Given a map ϕ from projections in $\mathcal{B}(H)$ to projections in $\mathcal{B}(H)$, let its *co-map* be the map

$$\psi : P \mapsto \bigvee \{Q : \phi(Q) \leq P\}.$$

Let $[A]$ denote the range projection of the operator $A \in \mathcal{B}(H)$.

Theorem 2.21. ([5], Theorem 4.4) *Let ϕ be a join preserving map from the set of projections in $\mathcal{B}(H)$ to itself such that $\phi(0) = 0$, and suppose that all of the projections in the ranges of ϕ and its co-map ψ commute. Then*

$$\phi(R) = \bigvee \{[AR] : A \in \mathcal{B}(H) \text{ and } \phi(Q)AQ = AQ \text{ for all projections } Q \in \mathcal{B}(H)\}$$

for every projection $R \in \mathcal{B}(H)$.

Proof. Let \mathcal{M} be a maximal abelian von Neumann algebra containing all of the projections in the ranges of ϕ and ψ , let $\Phi : L^\infty(X, \mu) \cong \mathcal{M}$ be an isomorphism, and define a measurable relation \mathcal{R} on X by setting $(p, q) \in \mathcal{R}$ if $\Phi(p)\phi(\Phi(q)) \neq 0$ (cf. Proposition 1.4). Let

$$\begin{aligned} \mathcal{V} &= \{A \in \mathcal{B}(H) : \phi(Q)AQ = AQ \text{ for all projections } Q \in \mathcal{B}(H)\} \\ &= \{A \in \mathcal{B}(H) : \phi(Q)A\psi(\phi(Q)) = A\psi(\phi(Q)) \text{ for all projections } Q \in \mathcal{B}(H)\} \\ &= \{A \in \mathcal{B}(H) : (p, q) \notin \mathcal{R} \Rightarrow \Phi(p)A\Phi(q) = 0\}. \end{aligned}$$

That is, $\mathcal{V} = \mathcal{V}_{\mathcal{R}}$ as in Theorem 2.9. Now fix a projection $R \in \mathcal{B}(H)$, let $P = \bigvee\{[AR] : A \in \mathcal{V}\} \leq \phi(R)$, and suppose $P < \phi(R)$. Then $\phi(R) \in \mathcal{M}$ by construction, and $\text{ran}(P)$ is invariant for \mathcal{M} so $P \in \mathcal{M}$ since \mathcal{M} is maximal abelian, so say $P = \Phi(p)$ and $\phi(R) = \Phi(r)$. Also let $Q = \bigvee\{[AR] : A \in \mathcal{M}\} \in \mathcal{M}$ and say $Q = \Phi(q)$. Now $(r - p, q) \in \mathcal{R}$ since $\phi(R) \leq \phi(Q)$, so Theorem 2.9 implies that there exists $A \in \mathcal{V}$ such that $\Phi(r - p)A\Phi(q) \neq 0$, contradicting the definition of P . We conclude that $P = \phi(R)$, as desired. \square

Theorem 4.4 of [5] is apparently more general than this since it covers maps from projections in $\mathcal{B}(H)$ to projections in $\mathcal{B}(K)$, but this version of the result follows easily from Theorem 2.21 by working in $\mathcal{B}(H \oplus K)$.

Finally, we have the following partially new result. It characterizes various classes of operator reflexive quantum relations over a maximal abelian von Neumann algebra.

Theorem 2.22. *Let \mathcal{M} be a maximal abelian von Neumann algebra in $\mathcal{B}(H)$, let $\Phi : L^\infty(X, \mu) \cong \mathcal{M}$ be an isomorphism, and let $\mathcal{V} \subseteq \mathcal{B}(H)$ be an operator reflexive operator space satisfying $\mathcal{M}\mathcal{V}\mathcal{M} \subseteq \mathcal{V}$. Then there is a measurable relation \mathcal{R} on X such that*

$$\mathcal{V} = \{A \in \mathcal{B}(H) : (p, q) \notin \mathcal{R} \Rightarrow \Phi(p)A\Phi(q) = 0\}.$$

If \mathcal{V} is a von Neumann algebra then \mathcal{R} is a measurable equivalence relation. If \mathcal{V} is an operator system then \mathcal{R} is a measurable graph. If \mathcal{V} is an operator algebra then \mathcal{R} is a measurable preorder. If \mathcal{V} is a triangular operator algebra then \mathcal{R} is a measurable partial order.

Proof. This follows from Proposition 2.16 together with the last part of Theorem 2.9, which implies that reflexivity, symmetry, antisymmetry, and transitivity of \mathcal{V} all carry over to $\mathcal{R}_{\mathcal{V}}$. \square

The converse assertions, that for any measurable relation (equivalence relation, graph, preorder, partial order) \mathcal{R} on X the set $\mathcal{V} = \{A \in \mathcal{B}(H) : (p, q) \notin \mathcal{R} \Rightarrow \Phi(p)A\Phi(q) = 0\}$ is an operator reflexive operator space (von Neumann algebra, operator system, operator algebra, triangular operator algebra) satisfying $\mathcal{M}\mathcal{V}\mathcal{M} \subseteq \mathcal{V}$, are trivial. So this gives us a complete characterization of these classes of operator reflexive operator bimodules over maximal abelian von Neumann algebras.

Theorem 2.22 reduces operator reflexive operator bimodules to various classes of measurable relations, but recall that we could reduce further to pointwise relations by Theorem 1.10.

2.6. Intrinsic characterization. Since quantum relations are effectively representation independent (Theorem 2.7), there should be an intrinsic characterization of them. We provide such a characterization in this section by axiomatizing the family of annihilating pairs of projections in $\mathcal{M} \otimes \mathcal{B}(\ell^2)$ introduced in Lemma 2.8.

First we note that in finite dimensions, quantum relations on \mathcal{M} naturally correspond to projections in $\mathcal{M} \otimes \mathcal{M}^{op}$. (We already know this when \mathcal{M} is atomic and abelian by Proposition 2.2, and we know it is false in the general abelian case by Theorem 2.9 and the discussion at the beginning of Section 1.3.)

Proposition 2.23. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a finite dimensional von Neumann algebra. Define an action Φ of $\mathcal{M} \otimes \mathcal{M}^{op}$ on $\mathcal{B}(H)$ by setting*

$$\Phi_{A \otimes C}(B) = ABC$$

for $A \in \mathcal{M}$, $C \in \mathcal{M}^{op}$, and $B \in \mathcal{B}(H)$ and extending linearly. Then for any quantum relation \mathcal{V} on \mathcal{M} the set

$$\mathcal{I}_{\mathcal{V}} = \{X \in \mathcal{M} \otimes \mathcal{M}^{op} : \Phi_X(B) = 0 \text{ for all } B \in \mathcal{V}\}$$

is a left ideal of $\mathcal{M} \otimes \mathcal{M}^{op}$, and for any left ideal \mathcal{I} of $\mathcal{M} \otimes \mathcal{M}^{op}$ the set

$$\mathcal{V}_{\mathcal{I}} = \{B \in \mathcal{B}(H) : \Phi_X(B) = 0 \text{ for all } X \in \mathcal{I}\}$$

is a quantum relation on \mathcal{M} . The two constructions are inverse to each other. The lattice of quantum relations on \mathcal{M} is order isomorphic to the lattice of projections in $\mathcal{M} \otimes \mathcal{M}^{op}$.

Proof. It is straightforward to check that $\mathcal{I}_{\mathcal{V}}$ is a left ideal and $\mathcal{V}_{\mathcal{I}}$ is a quantum relation. We verify that the two constructions are inverse to each other. By Theorem 2.7 we can choose the representation of \mathcal{M} , so take $H = \bigoplus \mathbf{C}^{n_i}$, $\mathcal{M} \cong \bigoplus_i M_{n_i}(\mathbf{C})$, and $\mathcal{M} \otimes \mathcal{M}^{op} \cong \bigoplus_{i,j} M_{n_i}(\mathbf{C}) \otimes M_{n_j}(\mathbf{C})^{op}$. Then the left ideals of $\mathcal{M} \otimes \mathcal{M}^{op}$ are all of the form $\bigoplus_{i,j} \mathcal{I}_{i,j}$ where $\mathcal{I}_{i,j}$ is a left ideal of $M_{n_i}(\mathbf{C}) \otimes M_{n_j}(\mathbf{C})^{op}$. The commutant of \mathcal{M} consists of the diagonal matrices that are constant on each \mathbf{C}^{n_i} , and consequently the bimodules over \mathcal{M}' are all of the form $\bigoplus_{i,j} \mathcal{V}_{i,j}$ where $\mathcal{V}_{i,j}$ is a subspace of $M_{n_i, n_j}(\mathbf{C})$. We now work in the summand corresponding to a single pair (i, j) .

The natural vector space isomorphism $M_{n_i, n_j}(\mathbf{C}) \cong \mathbf{C}^{n_i n_j}$ converts the action of $M_{n_i}(\mathbf{C}) \otimes M_{n_j}(\mathbf{C})^{op}$ to the standard action of $M_{n_i n_j}(\mathbf{C})$ in a way that is compatible with the natural isomorphism of $M_{n_i}(\mathbf{C}) \otimes M_{n_j}(\mathbf{C})^{op}$ with $M_{n_i n_j}(\mathbf{C})$, as can be seen by checking matrix units. So we reduce to showing that the map taking a left ideal of $M_k(\mathbf{C})$ to the subspace of \mathbf{C}^k it annihilates is inverse to the map taking a subspace of \mathbf{C}^k to the left ideal of $M_k(\mathbf{C})$ that annihilates it. This follows from the fact that the left ideals of $M_k(\mathbf{C})$ are all of the form $M_k(\mathbf{C})P$ for P a projection in $M_k(\mathbf{C})$ ([14], Theorem I.7.4).

This correspondence between quantum relations and left ideals is order reversing, but the map $(\mathcal{M} \otimes \mathcal{M}^{op})P \mapsto I - P$ is an order inverting 1-1 correspondence between the left ideals and the projections, so the lattice of quantum relations is naturally order isomorphic to the lattice of projections. \square

The main result of this section gives an intrinsic characterization of quantum relations over any von Neumann algebra. Recall that $[A]$ denotes the range projection of the operator A .

Definition 2.24. Let \mathcal{M} be a von Neumann algebra and let \mathcal{P} be the set of projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$, equipped with the restriction of the weak operator topology. An *intrinsic quantum relation* on \mathcal{M} is an open subset $\mathcal{R} \subset \mathcal{P} \times \mathcal{P}$ satisfying

- (i) $(0, 0) \notin \mathcal{R}$
- (ii) $(\bigvee P_{\lambda}, \bigvee Q_{\kappa}) \in \mathcal{R} \Leftrightarrow \text{some } (P_{\lambda}, Q_{\kappa}) \in \mathcal{R}$

$$(iii) (P, [BQ]) \in \mathcal{R} \Leftrightarrow ([B^*P], Q) \in \mathcal{R}$$

for all projections $P, Q, P_\lambda, Q_\kappa \in \mathcal{P}$ and all $B \in I \otimes \mathcal{B}(l^2)$.

This abstract version of quantum relations is helpful because some constructions become more natural when framed in these terms. Most significantly, this is true of the pullback construction described in part (b) of the following proposition. (On the other hand, some constructions are more natural in the concrete setting, for instance the product of quantum relations (Definition 2.4 (c)).)

Proposition 2.25. *Let \mathcal{M} and \mathcal{N} be von Neumann algebras.*

(a) *Any union of intrinsic quantum relations on \mathcal{M} is an intrinsic quantum relation on \mathcal{M} .*

(b) *If $\phi : \mathcal{M} \rightarrow \mathcal{N}$ is a unital weak* continuous *-homomorphism and \mathcal{R} is an intrinsic quantum relation on \mathcal{N} then*

$$\phi^*(\mathcal{R}) = \{(P, Q) : ((\phi \otimes \text{id})(P), (\phi \otimes \text{id})(Q)) \in \mathcal{R}\}$$

(with P and Q ranging over projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$) *is an intrinsic quantum relation on \mathcal{M} .*

The proof is straightforward. In part (b) we verify condition (iii) of Definition 2.24 using the identity

$$(\phi \otimes \text{id})([BQ]) = [(\phi \otimes \text{id})(BQ)] = [B(\phi \otimes \text{id})(Q)].$$

Pullbacks are not compatible with products. We already noted this in the atomic abelian case; see the comment following Definition 1.6.

As we mentioned in Section 2.4, the linking algebra construction allows us to embed any quantum relation in a quantum partial order, i.e., a weak* closed unital operator algebra, and for many purposes the two points of view do not substantively differ. However, pullbacks are clearly more natural in the quantum relation setting, as the pullback of a quantum partial order need not be a quantum partial order.

Before proceeding to the equivalence of Definitions 2.1 and 2.24 we give a nontrivial example. Let (X, μ) be a finitely decomposable measure space, let $\mathcal{M} \cong L^\infty(X, \mu)$ be the von Neumann algebra of bounded multiplication operators on $L^2(X, \mu)$, and let \mathcal{R} be a measurable relation on X . Recall the notion of \mathcal{R} -orthogonality for vectors in $L^2(X, \mu) \otimes l^2 \cong L^2(X; l^2)$ introduced in Section 2.4. Let $\tilde{\mathcal{R}}$ be the set of pairs of projections $P, Q \in \mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$ such that $h \not\perp_{\mathcal{R}} k$ for some $h \in \text{ran}(P)$ and $k \in \text{ran}(Q)$. We will now show that $\tilde{\mathcal{R}}$ is an intrinsic quantum relation. In fact, the quantum relation $\tilde{\mathcal{V}}_{\mathcal{R}}$ defined in Theorem 2.11 is the quantum relation associated to $\tilde{\mathcal{R}}$ according to the correspondence to be established in Theorem 2.32 below.

Lemma 2.26. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space (X, μ) and let $h, k, h_n, k_n \in L^2(X; l^2)$. Suppose that $h_n \rightarrow h$ and $k_n \rightarrow k$ in norm and that $h_n \perp_{\mathcal{R}} k_n$ for all n . Then $h \perp_{\mathcal{R}} k$.*

Proof. By passing to a subsequence, we can assume that $\|h - h_n\|, \|k - k_n\| \leq 1/n^2$ for all n . Now let $S, T \subseteq X$ and suppose $\epsilon = \inf\{|\langle k(y), h(x) \rangle| : x \in S, y \in T\} > 0$; we must show that $(\chi_S, \chi_T) \notin \mathcal{R}$. For each n let

$$S_n = \{x \in S : \|h(x) - h_n(x)\| \leq 1/n \text{ and } \|h_n(x)\| \leq n\epsilon/3\}$$

and let

$$T_n = \{y \in T : \|k(x) - k_n(x)\| \leq 1/n \text{ and } \|k(y)\| \leq n\epsilon/3\}.$$

Then

$$\begin{aligned} |\langle k_n(y), h_n(x) \rangle| &\geq |\langle k(y), h(x) \rangle| - |\langle k(y), h(x) - h_n(x) \rangle| - |\langle k(y) - k_n(y), h_n(x) \rangle| \\ &\geq \epsilon - \epsilon/3 - \epsilon/3 = \epsilon/3 \end{aligned}$$

for all $x \in S_n$ and $y \in T_n$, which implies that $(\chi_{S_n}, \chi_{T_n}) \notin \mathcal{R}$ since h_n is \mathcal{R} -orthogonal to k_n . Since $\|h - h_n\|, \|k - k_n\| \leq 1/n^2$, a simple computation shows that $S = \liminf S_n$ and $T = \liminf T_n$, and this yields $(\chi_S, \chi_T) \notin \mathcal{R}$, as desired. (See the proof of Lemma 1.12 for a detailed explication of this final step.) \square

Lemma 2.27. *Let \mathcal{R} be a measurable relation on a finitely decomposable measure space (X, μ) and let $h, h', k \in L^2(X; l^2)$. Suppose that $h \perp_{\mathcal{R}} k$ and $h' \perp_{\mathcal{R}} k$. Then $(h + h') \perp_{\mathcal{R}} k$.*

Proof. Let $\epsilon > 0$ and suppose $(\chi_S, \chi_T) \in \mathcal{R}$. Then for some $N \in \mathbf{N}$ we must have $(\chi_{S_N}, \chi_{T_N}) \in \mathcal{R}$ where $S_N = \{x \in S : \|h(x)\|, \|h'(x)\| \leq N\}$ and $T_N = \{y \in T : \|k(y)\| \leq N\}$. Now partition S_N and T_N into sets S_N^i and T_N^j such that $\|h(x) - h(x')\|, \|h'(x) - h'(x')\| \leq \epsilon/4N$ for all $x, x' \in S_N^i$ and $\|k(y) - k(y')\| \leq \epsilon/4N$ for all $y, y' \in T_N^j$. Then for some i and j we must have $(\chi_{S_N^i}, \chi_{T_N^j}) \in \mathcal{R}$, but this implies

$$\inf\{|\langle k(y), h(x) \rangle| : x \in S_N^i, y \in T_N^j\} = \inf\{|\langle k(y), h'(x) \rangle| : x \in S_N^i, y \in T_N^j\} = 0.$$

It follows that $|\langle k(y), h(x) \rangle|, |\langle k(y), h'(x) \rangle| \leq \epsilon/2$ for all $x \in S_N^i$ and $y \in T_N^j$, and hence $|\langle k(y), (h + h')(x) \rangle| \leq \epsilon$ for all $x \in S_N^i$ and $y \in T_N^j$. We conclude that

$$\inf\{|\langle k(y), (h + h')(x) \rangle| : x \in S, y \in T\} = 0.$$

This shows that $h + h'$ is \mathcal{R} -orthogonal to k . \square

Note that Lemma 2.27 also holds for sums in the second variable, since $h \perp_{\mathcal{R}} k$ if and only if $k \perp_{\mathcal{R}^T} h$, where \mathcal{R}^T is the transpose of \mathcal{R} (Definition 1.6 (b)).

Theorem 2.28. *Let (X, μ) be a finitely decomposable measure space, let $\mathcal{M} \cong L^\infty(X, \mu)$ be the von Neumann algebra of bounded multiplication operators on $L^2(X, \mu)$, and let \mathcal{R} be a measurable relation on X . Then the set $\tilde{\mathcal{R}}$ of pairs of projections $P, Q \in \mathcal{M} \bar{\otimes} \mathcal{B}(l^2)$ such that $h \not\perp_{\mathcal{R}} k$ for some $h \in \text{ran}(P)$ and $k \in \text{ran}(Q)$ is an intrinsic quantum relation on \mathcal{M} .*

Proof. First we check that the complement of $\tilde{\mathcal{R}}$ is closed. To see this suppose $P_\lambda \rightarrow P$ and $Q_\lambda \rightarrow Q$ and $(P_\lambda, Q_\lambda) \notin \tilde{\mathcal{R}}$ for all λ . Let $h \in \text{ran}(P)$ and $k \in \text{ran}(Q)$. Since the weak and strong operator topologies agree on \mathcal{P} , for any $n \in \mathbf{N}$ there exists λ and $h_n \in \text{ran}(P_\lambda)$, $k_n \in \text{ran}(Q_\lambda)$ such that $\|h_n - h\|, \|k_n - k\| \leq 1/n$. Lemma 2.26 therefore yields $h \perp_{\mathcal{R}} k$, and we conclude that $(P, Q) \notin \tilde{\mathcal{R}}$.

Next, it is clear that $(0, 0) \notin \tilde{\mathcal{R}}$ and that $P' \leq P$, $Q' \leq Q$, $(P', Q') \in \tilde{\mathcal{R}}$ implies $(P, Q) \in \tilde{\mathcal{R}}$. Now suppose $(P_\lambda, Q_\kappa) \notin \tilde{\mathcal{R}}$ for all λ, κ . By a double application of Lemma 2.27 we have $h \perp_{\mathcal{R}} k$ for any h in the unclosed sum of the ranges of the P_λ and any k in the unclosed sum of the ranges of the Q_κ , and $(\bigvee P_\lambda, \bigvee Q_\kappa) \notin \tilde{\mathcal{R}}$ then follows from Lemma 2.26. This verifies condition (ii) of Definition 2.24.

Finally, let P and Q be projections in $\mathcal{M} \bar{\otimes} \mathcal{B}(l^2)$ and let $B \in I \otimes \mathcal{B}(l^2)$. Lemma 2.26 implies that $(P, [BQ]) \in \tilde{\mathcal{R}}$ if and only if $h \not\perp_{\mathcal{R}} Bk$ for some $h \in \text{ran}(P)$ and $k \in \text{ran}(Q)$. Writing $B = I \otimes B_0$, we have $(B^*h)(x) = B_0^*(h(x))$ and $(Bk)(y) = B_0(k(y))$ for all x and y , so that

$$\langle (Bk)(y), h(x) \rangle = \langle B_0(k(y)), h(x) \rangle = \langle k(y), B_0^*(h(x)) \rangle = \langle k(y), (B^*h)(x) \rangle.$$

It follows that $h \not\prec_{\mathcal{R}} Bk$ for some $h \in \text{ran}(P)$ and $k \in \text{ran}(Q)$ if and only if $B^*h \not\prec_{\mathcal{R}} k$ for some such h and k , and another application of Lemma 2.26 shows that this is equivalent to $([B^*P], Q) \in \tilde{\mathcal{R}}$. This verifies condition (iii) of Definition 2.24. \square

We now begin preparing for Theorem 2.32, which intrinsically characterizes quantum relations. We first collect some easy consequences of Definition 2.24.

Lemma 2.29. *Let \mathcal{R} be an intrinsic quantum relation on a von Neumann algebra \mathcal{M} .*

- (a) *For any projections P and Q in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$ we have $(P, 0), (0, Q) \notin \mathcal{R}$.*
- (b) *If P and Q are projections in $I \otimes \mathcal{B}(l^2)$ and $PQ = 0$ then $(P, Q) \notin \mathcal{R}$.*
- (c) *If $B \in I \otimes \mathcal{B}(l^2)$ is an isometry then*

$$(P, Q) \in \mathcal{R} \iff (BPB^*, BQB^*) \in \mathcal{R}$$

for any projections P and Q in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$.

- (d) *If P and Q are projections in $I \otimes \mathcal{B}(l^2)$ with orthogonal ranges and P_1, P_2, Q_1, Q_2 are projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$ satisfying $P_1, P_2 \leq P$ and $Q_1, Q_2 \leq Q$ then*

$$(P_1 + Q_1, P_2 + Q_2) \in \mathcal{R} \iff (P_1, P_2) \in \mathcal{R} \text{ or } (Q_1, Q_2) \in \mathcal{R}.$$

Proof. (a) Since $(0, 0) \notin \mathcal{R}$, this follows by taking $B = Q = 0$ or $B = P = 0$ in condition (iii) of Definition 2.24.

- (b) Let $B = Q$; then $B^*P = 0$ and $BQ = Q$, and so

$$(P, Q) = (P, [BQ]) \in \mathcal{R} \iff (0, Q) = ([B^*P], Q) \in \mathcal{R}.$$

But $(0, Q) \notin \mathcal{R}$ by part (a), so $(P, Q) \notin \mathcal{R}$.

- (c) We have $[B^*(BPB^*)] = [PB^*] = P$ and $[BQ] = BQB^*$, so

$$\begin{aligned} (P, Q) \in \mathcal{R} &\iff ([B^*(BPB^*)], Q) \in \mathcal{R} \\ &\iff (BPB^*, [BQ]) \in \mathcal{R} \\ &\iff (BPB^*, BQB^*) \in \mathcal{R} \end{aligned}$$

by condition (iii) of Definition 2.24.

(d) By part (b) we have $(P, Q), (Q, P) \notin \mathcal{R}$. Since $P_1, P_2 \leq P$ and $Q_1, Q_2 \leq Q$, this implies that $(P_1, Q_2), (Q_1, P_2) \notin \mathcal{R}$ (using condition (ii) of Definition 2.24). Now since $P_1 + Q_1 = P_1 \vee Q_1$ and $P_2 + Q_2 = P_2 \vee Q_2$, condition (ii) of Definition 2.24 implies that $(P_1 + Q_1, P_2 + Q_2) \in \mathcal{R}$ if and only if at least one of (P_1, P_2) , (P_1, Q_2) , (Q_1, P_2) , or (Q_1, Q_2) belongs to \mathcal{R} . As (P_1, Q_2) and (Q_1, P_2) cannot belong to \mathcal{R} , the desired conclusion follows. \square

We introduce the following temporary notation. For any Hilbert space H and any vectors $v, w \in H \otimes l^2$, let $\omega_{v,w}$ be the weak* continuous linear functional on $\mathcal{B}(H)$ defined by $\omega_{v,w}(A) = \langle (A \otimes I)w, v \rangle$. Also, given a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$, for any $v \in H \otimes l^2$ let $P_{[v]}$ be the smallest projection in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$ whose range contains v .

Lemma 2.30. *Let \mathcal{R} be an intrinsic quantum relation on $\mathcal{M} \subseteq \mathcal{B}(H)$ and suppose $v, w, v', w' \in H \otimes l^2$ satisfy $\omega_{v,w} = \omega_{v',w'}$. Also assume $v - v', w - w' \in H \otimes K_0$ for some finite dimensional subspace K_0 of l^2 . Then $(P_{[v]}, P_{[w]}) \in \mathcal{R}$ if and only if $(P_{[v']}, P_{[w']}) \in \mathcal{R}$.*

Proof. First, let B_0 be an isometry from l^2 onto an infinite-codimensional subspace of l^2 , let $B = I \otimes B_0$, and replace v, w, v', w' with Bv, Bw, Bv', Bw' . We can do this because

$$\omega_{Bv, Bw} = \omega_{v, w} = \omega_{v', w'} = \omega_{Bv', Bw'},$$

and

$$(P_{[v]}, P_{[w]}) \in \mathcal{R} \iff (P_{[Bv]}, P_{[Bw]}) = (BP_{[v]}B^*, BP_{[w]}B^*) \in \mathcal{R}$$

by part (c) of Lemma 2.29 (and similarly for v', w'). Also replace K_0 with $B_0(K_0)$.

Now let $P_0 \in \mathcal{B}(l^2)$ be the orthogonal projection onto K_0 . Then $(I \otimes P_0)v$ belongs to $H \otimes K_0$ and hence (since K_0 is finite dimensional) is a finite linear combination of elementary tensors. The same is true of w, v' , and w' . So there is a finite dimensional subspace H_0 of H such that the projections of all four vectors onto $H \otimes K_0$ lie in $H_0 \otimes K_0$ and $v - v', w - w' \in H_0 \otimes K_0$.

If K_1 is a finite dimensional subspace of l^2 that is orthogonal to the range of B_0 then all four vectors are orthogonal to $H \otimes K_1$. In the remainder of the proof we will work in the finite dimensional space $H_0 \otimes (K_0 \oplus K_1)$, where K_1 is chosen large enough to accomodate all computations below. (Specifically, $\dim(K_1) = (\dim(H_0) - 1)^2 \cdot \dim(K_0) + 1$ would suffice, but this number is not important.)

Identify H_0 with \mathbf{C}^k , $K_0 \oplus K_1$ with \mathbf{C}^n , and $H_0 \otimes (K_0 \oplus K_1)$ with $\mathbf{C}^k \oplus \cdots \oplus \mathbf{C}^k$ (n summands). Let (e_i) be the standard basis for \mathbf{C}^k and let $\bar{v} = (I \otimes P_0)v$, $\bar{w} = (I \otimes P_0)w$, $\bar{v}' = (I \otimes P_0)v'$, and $\bar{w}' = (I \otimes P_0)w'$. The main step is to incrementally convert \bar{v} and \bar{w} , which initially lie in $H_0 \otimes K_0$, into vectors of the form $a_1 e_{i_1} \oplus \cdots \oplus a_n e_{i_n}$ and $b_1 e_{j_1} \oplus \cdots \oplus b_n e_{j_n}$, now lying in $H_0 \otimes (K_0 \oplus K_1)$, without changing $\omega_{v, w}$ or affecting whether $(P_{[v]}, P_{[w]})$ lies in \mathcal{R} , and similarly to put \bar{v}' and \bar{w}' in the form $a'_1 e_{i'_1} \oplus \cdots \oplus a'_n e_{i'_n}$ and $b'_1 e_{j'_1} \oplus \cdots \oplus b'_n e_{j'_n}$.

The main step is achieved in the following way. Say $\bar{v} = v_1 \oplus \cdots \oplus v_n$ with each $v_i \in H_0 \cong \mathbf{C}^k$ and suppose some v_r is not of the form $a_r e_{i_r}$. Let the corresponding decomposition of \bar{w} be $\bar{w} = w_1 \oplus \cdots \oplus w_n$. Because the dimension of K_1 was sufficiently large, some of the v_i 's and w_i 's are zero regardless of where we are in the construction. For notational simplicity say $v_1 = w_1 = 0$ and $v_2 = a'_1 e_1 + a''_1 u$ with $a'_1 e_1 \neq 0$, $a''_1 u \neq 0$, and $u \perp e_1$. Now consider the vectors

$$v^0 = (a'_1 e_1 - a''_1 u) \oplus (a'_1 e_1 + a''_1 u) \oplus v_3 \oplus \cdots \oplus v_n \oplus \tilde{v}$$

and

$$v^1 = (-a'_1 e_1 + a''_1 u) \oplus (a'_1 e_1 + a''_1 u) \oplus v_3 \oplus \cdots \oplus v_n \oplus \tilde{v},$$

where $\tilde{v} = v - \bar{v} \in (H \otimes (K_0 \oplus K_1))^\perp$. We have

$$(P_{[v^0]}, P_{[w]}) \in \mathcal{R} \iff (P_{[v^1]}, P_{[w]}) \in \mathcal{R}$$

since $v^0 = Uv^1$ and $w = Uw$, and hence $P_{[v^1]} = U^*P_{[v^0]}U$ and $P_{[w]} = U^*P_{[w]}U$, where $U = -I_H \oplus I_H \oplus \cdots \oplus I_H \oplus I_H \in I \otimes \mathcal{B}(l^2)$. Hence both pairs belong to \mathcal{R} if and only if $(P_{[v^0]} \vee P_{[v^1]}, P_{[w]}) \in \mathcal{R}$. But $P_{[v^0]} \vee P_{[v^1]} = P_{[v]} \vee P_{[\tilde{v}]}$ where $\tilde{v} = (a'_1 e_1 - a''_1 u) \oplus 0 \oplus \cdots \oplus 0 \oplus 0$, and we have $(P_{[\tilde{v}]}, P_{[w]}) \notin \mathcal{R}$ by Lemma 2.29 (d). Thus

$$(P_{[v^0]}, P_{[w]}) \in \mathcal{R} \iff (P_{[v]}, P_{[w]}) \in \mathcal{R}.$$

Also $\omega_{v, w} = \omega_{v^0, w}$. So replace v with v^0 and then apply the unitary

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \oplus I \oplus \cdots \oplus I \oplus I \in I \otimes B(l^2)$$

to v and w so that \bar{v} becomes $\sqrt{2}a'_1e_1 \oplus \sqrt{2}a''_1u \oplus v_3 \oplus \cdots \oplus v_r$ and \bar{w} becomes $\frac{1}{\sqrt{2}}w_2 \oplus \frac{1}{\sqrt{2}}w_2 \oplus w_3 \oplus \cdots \oplus w_n$. The end result is that \bar{v} has moved one step closer to being in the desired form, $\omega_{v,w}$ has not changed, and whether $(P_{[v]}, P_{[w]}) \in \mathcal{R}$ has not changed. The vector w has also been replaced by Vw , but if \bar{w} was already in the desired form this will still be the case. So we achieve the main step by first putting \bar{w} in the desired form and then putting \bar{v} in the desired form.

Now we proceed in four additional steps. As above let $\bar{v} = v_1 \oplus \cdots \oplus v_n$ and $\bar{w} = w_1 \oplus \cdots \oplus w_n$. We can also write $v_r = a_re_{i_r}$ and $w_r = b_re_{j_r}$ for $1 \leq r \leq n$. First, for any r , if $v_r = 0$ then we also set $w_r = 0$ and if $w_r = 0$ then we also set $v_r = 0$. This clearly does not change $\omega_{v,w}$; to see that it also does not affect whether $(P_{[v]}, P_{[w]}) \in \mathcal{R}$, suppose for simplicity that $w_1 = 0$ and consider the vector $v^1 = Uv$ where $U = -I \oplus I \oplus \cdots \oplus I \oplus I$ as above. Letting $v^0 = v$ and arguing exactly as in the main step yields the desired conclusion. (We do not need to apply V for this argument.) Make the same argument for v' and w' .

We now have $v_rw_r \neq 0$ or $v_r = w_r = 0$ for all r . The next step eliminates duplications where $i_r = i_s$ and $j_r = j_s$ but $r \neq s$ (and a_r, b_r, a_s, b_s are all nonzero). To do this, for notational simplicity suppose $r = 1$ and $s = 2$ and apply a unitary of the form

$$\begin{bmatrix} \alpha I & \beta I \\ -\bar{\beta} I & \bar{\alpha} I \end{bmatrix} \oplus I \oplus \cdots \oplus I \oplus I \in I \otimes B(l^2)$$

to v and w with α and β chosen so that $\alpha a_1 + \beta a_2 = 0$. This leads to $v_1 = 0$ and $v_2 = (-\bar{\beta}a_1 + \bar{\alpha}a_2)e_{i_2}$. We may have $w_1 \neq 0$ but the argument of the previous step can now be repeated to remedy this. Applying the preceding construction repeatedly, we reach a point where $r \neq s$ and v_r, w_r, v_s, w_s all nonzero implies either $i_r \neq i_s$ or $j_r \neq j_s$. Make the same argument for v' and w' .

In the next step we leave v' and w' intact and apply a unitary in $I \otimes \mathcal{B}(l^2)$ to v and w to ensure $i_r = i'_r$ and $j_r = j'_r$ for all r such that $v_r, w_r \neq 0$. We just use a permutation unitary to achieve this; the pairs (i_r, j_r) appearing in nonzero components of v and w are the same up to rearrangement as the pairs (i'_r, j'_r) appearing in nonzero components of v' and w' since $\omega_{v,w} = \omega_{v',w'}$. This is because applying $\omega_{v,w}$ to the operator $V_{e_{i_r}e_{j_r}} : u \mapsto \langle u, e_{j_r} \rangle e_{i_r}$ in $\mathcal{B}(H)$ yields the result $\bar{a}_r b_r$, so we can diagnose whether (i_r, j_r) appears in a nonzero component of (v, w) in this way.

We have reached the final step. By applying both sides of $\omega_{v,w} = \omega_{v',w'}$ to $V_{e_{i_r}e_{j_r}}$ we see that $a_r \bar{b}_r = a'_r \bar{b}'_r$ for all values of r such that $v_r, w_r, \bar{v}_r, \bar{w}_r$ are nonzero, so let

$$B = \frac{b_1}{b'_1} I \oplus \cdots \oplus \frac{b_n}{b'_n} I \oplus I \in I \otimes \mathcal{B}(l^2)$$

(with the convention that $\frac{0}{0} = 1$, so that B is invertible) and observe that

$$(P_{[v]}, P_{[w]}) = (P_{[v]}, [BP_{[w']}]) \in \mathcal{R} \quad \Leftrightarrow \quad (P_{[v']}, P_{[w']}) = ([B^*P_{[v]}], P_{[w']}) \in \mathcal{R}.$$

Since the truth values of the conditions $(P_{[v]}, P_{[w]}) \in \mathcal{R}$ and $(P_{[v']}, P_{[w']}) \in \mathcal{R}$ have not changed throughout the entire process we conclude that $(P_{[v]}, P_{[w]}) \in \mathcal{R} \Leftrightarrow (P_{[v']}, P_{[w']}) \in \mathcal{R}$ for the original values of v, w, v', w' . \square

Let $P_S(A)$ denote the spectral projection of a self-adjoint operator A for the Borel set $S \subseteq \mathbf{R}$.

Lemma 2.31. *Let $\{A_\lambda\}$ be a bounded net of self-adjoint elements of a von Neumann algebra \mathcal{M} and suppose $A_\lambda \rightarrow A$ weak operator. Then for any $\epsilon > 0$ there is*

a net of projections $\{P_\kappa\}$ in \mathcal{M} which converges weak operator to $P_{(-\infty,0]}(A)$ and such that every P_κ is less than or equal to $P_{(-\infty,\epsilon]}(A_\lambda)$ for some λ .

Proof. Fix $\epsilon > 0$. Then let $\delta > 0$, let v_1, \dots, v_m be unit vectors in $\text{ran}(P_{(-\infty,0]}(A))$, and let w_1, \dots, w_n be unit vectors in $\text{ran}(P_{(0,\infty)}(A))$. It will suffice to find a projection $P \leq P_{(-\infty,\epsilon]}(A_\lambda)$ in \mathcal{M} for some λ such that (1) $\|Pv_i\|^2 \geq 1 - \delta$ for all i and (2) $\|Pw_j\|^2 \leq \delta$ for all j . We will achieve this with a projection of the form $P = P_{[1/2,1]}((QRQ)^n)$ where $Q = P_{(-\infty,\alpha]}(A_\lambda)$, for some λ and some $\alpha \leq \epsilon$, and $R = P_{(-\infty,0]}(A)$. It is easy to see that any such P belongs to \mathcal{M} and satisfies $P \leq P_{(-\infty,\epsilon]}(A_\lambda)$.

We first check that property (2) can be assured independently of the choice of α and λ simply by choosing n large enough. Since $P \leq 2(QRQ)^n$, this follows from the following claim: if Q and R are any projections and $Rw = 0$ then

$$\|(RQ)^n w\| \leq \frac{1}{\sqrt{2n-1}} \|w\|$$

(for $n \geq 1$). This can be seen by using the general form of two projections given in [14], p. 308. Namely, we can decompose the Hilbert space so that

$$R = \begin{bmatrix} R_0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_0 & 0 & 0 \\ 0 & C^2 & CS \\ 0 & CS & S^2 \end{bmatrix}$$

with R_0 and Q_0 commuting projections, $0 \leq C, S \leq I$, and $C^2 + S^2 = I$. For then $Rw = 0$ means that w has the form $w = \begin{bmatrix} w' \\ 0 \\ w'' \end{bmatrix}$ with $R_0 w' = 0$ and hence that

$(RQ)^n w = \begin{bmatrix} 0 \\ C^{2n-1} S w'' \\ 0 \end{bmatrix}$. So we just need to estimate the norm of $C^{2n-1} S w'' = C(I - S^2)^{n-1} S w''$. But if S is realized as multiplication by x then $(I - S^2)^{n-1} S$ becomes multiplication by $x(1-x^2)^{n-1}$, which is extremized on $[0, 1]$ at $x = \pm \frac{1}{\sqrt{2n-1}}$ and hence has operator norm at most $\frac{1}{\sqrt{2n-1}}$ (ignoring the $(1-x^2)^{n-1}$ factor, which is at most one). Thus $\|C^{2n-1} S w''\| \leq \frac{1}{\sqrt{2n-1}} \|w\|$ and this completes the proof of the claim.

So fix a value of n that ensures property (2). For property (1), let $\alpha = \min\{\delta/2(n+1), \epsilon\}$ and choose λ so that $\langle A_\lambda v_i, v_i \rangle \leq \alpha^3$ for $1 \leq i \leq m$. Then set $Q = P_{(-\infty,\alpha]}(A_\lambda)$, $R = P_{(-\infty,0]}(A)$, and $P = P_{[1/2,1]}((QRQ)^n)$. We must verify that $\|Pv_i\|^2 \geq 1 - \delta$ for all i . First, we have

$$\alpha \langle (I - Q)v_i, v_i \rangle \leq \langle A_\lambda v_i, v_i \rangle \leq \alpha^3$$

so that

$$\|(I - Q)v_i\|^2 = \langle (I - Q)v_i, v_i \rangle \leq \alpha^2,$$

i.e., $\|(I - Q)v_i\| \leq \alpha$. Also $(I - R)v_i = 0$, so

$$\begin{aligned} \|v_i - (QRQ)^n v_i\| &\leq \|v_i - Qv_i\| + \|Qv_i - QRv_i\| + \dots + \|(QR)^n v_i - (QR)^n Qv_i\| \\ &\leq \|(I - Q)v_i\| + \|Q\| \|(I - R)v_i\| + \dots + \|(QR)^n\| \|(I - Q)v_i\| \\ &\leq (n+1)\alpha \\ &\leq \delta/2 \end{aligned}$$

and hence

$$\begin{aligned}
\|(I - P)v_i\|^2 &= \|P_{[0,1/2)}((QRQ)^n)v_i\|^2 \\
&= \|P_{(1/2,1]}(I - (QRQ)^n)v_i\|^2 \\
&= \langle P_{(1/2,1]}(I - (QRQ)^n)v_i, v_i \rangle \\
&\leq 2\langle (I - (QRQ)^n)v_i, v_i \rangle \\
&\leq \delta.
\end{aligned}$$

This shows that $\|Pv_i\|^2 \geq 1 - \delta$, as desired. \square

Theorem 2.32. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let \mathcal{P} be the set of projections in $\mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$. If \mathcal{V} is a quantum relation on \mathcal{M} (Definition 2.1) then*

$$\mathcal{R}_{\mathcal{V}} = \{(P, Q) \in \mathcal{P}^2 : P(A \otimes I)Q \neq 0 \text{ for some } A \in \mathcal{V}\}$$

is an intrinsic quantum relation on \mathcal{M} (Definition 2.24); conversely, if \mathcal{R} is an intrinsic quantum relation on \mathcal{M} then

$$\mathcal{V}_{\mathcal{R}} = \{A \in \mathcal{B}(H) : (P, Q) \notin \mathcal{R} \Rightarrow P(A \otimes I)Q = 0\}$$

is a quantum relation on \mathcal{M} . The two constructions are inverse to each other.

Proof. Observe first that

$$\begin{aligned}
P(A \otimes I)Q = 0 &\Leftrightarrow \langle (A \otimes I)w, v \rangle = 0 \text{ for all } v \in \text{ran}(P), w \in \text{ran}(Q) \\
&\Leftrightarrow \omega_{v,w}(A) = 0 \text{ for all } v \in \text{ran}(P), w \in \text{ran}(Q).
\end{aligned}$$

Now let \mathcal{V} be a quantum relation on \mathcal{M} . Then conditions (i) and (ii) of Definition 2.24 are easily seen to hold for $\mathcal{R}_{\mathcal{V}}$, and condition (iii) holds because $B \in I \otimes \mathcal{B}(l^2)$ implies

$$\begin{aligned}
P(A \otimes I)[BQ] \neq 0 &\Leftrightarrow P(A \otimes I)BQ \neq 0 \\
&\Leftrightarrow PB(A \otimes I)Q \neq 0 \\
&\Leftrightarrow [B^*P](A \otimes I)Q \neq 0.
\end{aligned}$$

Also, using the fact that the weak operator topology agrees with the strong operator topology on \mathcal{P} it is easy to see that the complement of $\mathcal{R}_{\mathcal{V}}$ is closed in \mathcal{P}^2 . So $\mathcal{R}_{\mathcal{V}}$ is an intrinsic quantum relation.

Next let \mathcal{R} be an intrinsic quantum relation on \mathcal{M} . It is clear that $\mathcal{V}_{\mathcal{R}}$ is a linear subspace of $\mathcal{B}(H)$, it is weak* closed by the observation made at the start of the proof, and it is a bimodule over \mathcal{M}' because if $A, C \in \mathcal{M}'$, $B \in \mathcal{V}_{\mathcal{R}}$, and $P, Q \in \mathcal{P}$ then $P(B \otimes I)Q = 0$ implies

$$P(ABC \otimes I)Q = P(A \otimes I)(B \otimes I)(C \otimes I)Q = (A \otimes I)P(B \otimes I)Q(C \otimes I) = 0.$$

So $\mathcal{V}_{\mathcal{R}}$ is a quantum relation.

Now let \mathcal{V} be a quantum relation, let $\mathcal{R} = \mathcal{R}_{\mathcal{V}}$, and let $\tilde{\mathcal{V}} = \mathcal{V}_{\mathcal{R}}$. Then it is immediate that $\mathcal{V} \subseteq \tilde{\mathcal{V}}$, and the reverse inclusion is just the content of Lemma 2.8.

Finally, let \mathcal{R} be an intrinsic quantum relation, let $\mathcal{V} = \mathcal{V}_{\mathcal{R}}$, and let $\tilde{\mathcal{R}} = \mathcal{R}_{\mathcal{V}}$. It is immediate that $\tilde{\mathcal{R}} \subseteq \mathcal{R}$. For the reverse inclusion, fix P and Q and suppose $P(A \otimes I)Q = 0$ for all $A \in \mathcal{V}$; we must show that $(P, Q) \notin \mathcal{R}$. By condition (ii) of Definition 2.24 it will suffice to show that $(P_{[v]}, P_{[w]}) \notin \mathcal{R}$ for any $v \in \text{ran}(P)$ and $w \in \text{ran}(Q)$.

Let $E \subseteq \mathcal{B}(H)_*$ be the norm closure of $\{\omega_{v,w} : (P_{[v]}, P_{[w]}) \notin \mathcal{R}\}$. We claim that E is a linear subspace. To see this, suppose $(P_{[v_1]}, P_{[w_1]}), (P_{[v_2]}, P_{[w_2]}) \notin \mathcal{R}$ and let $V_1, V_2 \in I \otimes \mathcal{B}(l^2)$ be isometries with orthogonal ranges; then $v = V_1v_1 + V_2v_2$ and $w = V_1w_1 + V_2w_2$ satisfy $P_{[v]} \leq V_1P_{[v_1]}V_1^* + V_2P_{[v_2]}V_2^*$ and $P_{[w]} \leq V_1P_{[w_1]}V_1^* + V_2P_{[w_2]}V_2^*$.

$V_2 P_{[w_2]} V_2^*$, and hence $(P_{[v]}, P_{[w]}) \notin \mathcal{R}$ by Lemma 2.29 (c) and (d). But $\omega_{v,w} = \omega_{v_1,w_1} + \omega_{v_2,w_2}$, so we have shown that E is stable under addition. Stability under scalar multiplication is easy. This proves the claim.

Now let $v \in \text{ran}(P)$ and $w \in \text{ran}(Q)$; we must show that $(P_{[v]}, P_{[w]}) \notin \mathcal{R}$. We may suppose $\|v\| = \|w\| = 1$. If $\omega_{v,w} \notin E$ then there would exist $A \in \mathcal{B}(H)$ such that $\omega_{v,w}(A) \neq 0$ but $\omega(A) = 0$ for all $\omega \in E$; then $A \in \mathcal{V}$ but $P_{[v]}(A \otimes I)P_{[w]} \neq 0$, contradicting the fact that $P_{[v]} \subseteq P$ and $P_{[w]} \subseteq Q$. Thus $\omega_{v,w} \in E$. We conclude the proof by showing that any unit vectors $v, w \in H \otimes l^2$ with $\omega_{v,w} \in E$ satisfy $(P_{[v]}, P_{[w]}) \notin \mathcal{R}$.

Let $B_0 \in \mathcal{B}(l^2)$ be an isometry with infinite codimensional range and let $B = I \otimes B_0$. We may replace v and w with Bv and Bw since $\omega_{v,w} = \omega_{Bv,Bw}$ and $(P_{[v]}, P_{[w]}) \in \mathcal{R} \Leftrightarrow (P_{[Bv]}, P_{[Bw]}) = (BP_{[v]}B^*, BP_{[w]}B^*) \in \mathcal{R}$. Now since $\omega_{v,w} \in E$, for any $0 < \epsilon \leq 1$ there exist $v', w' \in H \otimes l^2$ such that $\|\omega_{v,w} - \omega_{v',w'}\| \leq \epsilon$ and $(P_{[v']}, P_{[w']}) \notin \mathcal{R}$. We may also replace v' and w' with Bv' and Bw' . Find a finite rank projection $R_0 \in \mathcal{B}(l^2)$ such that $\|(I - R)v\|, \|(I - R)w\|, \|(I - R)v'\|, \|(I - R)w'\| \leq \epsilon$ where $R = I \otimes R_0$, and let $v_1 = Rv + (I - R)v'$ and $w_1 = Rw + (I - R)w'$. Then $\|v - v_1\|, \|w - w_1\| \leq 2\epsilon$,

$$\begin{aligned} \|\omega_{v_1,w_1} - \omega_{v',w'}\| &\leq \|\omega_{v_1,w_1} - \omega_{v_1,w}\| + \|\omega_{v_1,w} - \omega_{v,w}\| + \|\omega_{v,w} - \omega_{v',w'}\| \\ &\leq \|v_1\| \|w_1 - w\| + \|v_1 - v\| \|w\| + \epsilon \\ &\leq (1 + \epsilon) \cdot 2\epsilon + 2\epsilon + \epsilon \\ &\leq 7\epsilon, \end{aligned}$$

and $v_1 - v', w_1 - w' \in H \otimes K_0$ where $K_0 = \text{ran}(R_0)$. This implies that $\omega_{v_1,w_1} - \omega_{v',w'} = \text{tr}(\cdot A)$ for some finite rank operator A with $\text{tr}(|A|) \leq 7\epsilon$. Thus $\omega_{v_1,w_1} - \omega_{v',w'} = \omega_{v_2,w_2}$ for some vectors $v_2, w_2 \in H \otimes K_1$ such that $\|v_2\|, \|w_2\| \leq \sqrt{7\epsilon}$ and where K_1 is a finite dimensional subspace of l^2 that we can take to be orthogonal to $\text{ran}(B_0)$. Finally let $v_3 = v_1 - v_2$ and $w_3 = w_1 - w_2$. We obtain $\|v - v_3\|, \|w - w_3\| \leq 2\epsilon + \sqrt{7\epsilon}$, $\omega_{v_3,w_3} = \omega_{v',w'}$, and $v_3 - v', w_3 - w' \in H \otimes (K_0 \oplus K_1)$. Since $(P_{[v']}, P_{[w']}) \notin \mathcal{R}$, Lemma 2.30 implies that $(P_{[v_3]}, P_{[w_3]}) \notin \mathcal{R}$.

Letting $\epsilon \rightarrow 0$, we thus get a sequence of pairs of projections $P_n, Q_n \in \mathcal{P}$ such that $(P_n, Q_n) \notin \mathcal{R}$ and $\|P_n v\|, \|Q_n w\| \rightarrow 1$. Passing to a weak operator convergent subnet $\{P_\lambda \oplus Q_\lambda\}$ of the sequence $\{P_n \oplus Q_n\}$, we have $(I - P_\lambda) \oplus (I - Q_\lambda) \rightarrow A_1 \oplus A_2$ weak operator for some positive $A_1, A_2 \in \mathcal{M} \overline{\otimes} \mathcal{B}(l^2)$ such that $v \in \ker(A_1)$ and $w \in \ker(A_2)$. By Lemma 2.31 with $\epsilon = 1/2$ we can find a net of projections $P'_\lambda \oplus Q'_\lambda \leq P_\lambda \oplus Q_\lambda$ which converge weak operator to $\tilde{P} \oplus \tilde{Q}$ where \tilde{P} and \tilde{Q} are the orthogonal projections onto $\ker(A_1)$ and $\ker(A_2)$, respectively. So $(\tilde{P}, \tilde{Q}) \notin \mathcal{R}$ since \mathcal{R} is open, and then $P_{[v]} \leq \tilde{P}$, $P_{[w]} \leq \tilde{Q}$ implies $(P_{[v]}, P_{[w]}) \notin \mathcal{R}$. This is what we needed to prove. \square

2.7. Quantum tori. In this section we analyze quantum relations on quantum tori which satisfy a condition that is naturally understood as “translation invariance”. We find that this class of quantum relations is quite tractable.

Quantum tori are the simplest examples of noncommutative manifolds. They are related to the quantum plane, which plays the role of the phase space of a spinless one-dimensional particle. The classical version of such a system has phase space \mathbf{R}^2 , with the point $(q, p) \in \mathbf{R}^2$ representing a state with position q and momentum p , so that the position and momentum observables are just the coordinate functions on phase space. When such a system is quantized the position and momentum observables are modelled by unbounded self-adjoint operators Q and P satisfying

$QP - PQ = i\hbar I$. Polynomials in Q and P can then be seen as a quantum analog of polynomial functions on \mathbf{R}^2 . The quantum analog of the continuous functions on the torus — equivalently, the $(2\pi, 2\pi)$ -periodic continuous functions on the plane — is the C^* -algebra generated by the unitary operators e^{iQ} and e^{iP} , which satisfy the commutation relation $e^{iQ}e^{iP} = e^{-i\hbar}e^{iP}e^{iQ}$. For more background see [13] or Sections 4.1, 4.2, 5.5, and 6.6 of [20].

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ and fix $\hbar \in \mathbf{R}$. Let $\{e_{m,n}\}$ be the standard basis of $l^2(\mathbf{Z}^2)$. We model the quantum tori on $l^2(\mathbf{Z}^2)$ as follows.

Definition 2.33. Let U_\hbar and V_\hbar be the unitaries in $\mathcal{B}(l^2(\mathbf{Z}^2))$ defined by

$$\begin{aligned} U_\hbar e_{m,n} &= e^{-i\hbar n/2} e_{m+1,n} \\ V_\hbar e_{m,n} &= e^{i\hbar m/2} e_{m,n+1}. \end{aligned}$$

The *quantum torus von Neumann algebra* for the given value of \hbar is the von Neumann algebra $W^*(U_\hbar, V_\hbar)$ generated by U_\hbar and V_\hbar .

If \hbar is an irrational multiple of π then $W^*(U_\hbar, V_\hbar)$ is a hyperfinite II_1 factor. We will not need this fact.

Conjugating U_\hbar and V_\hbar by the Fourier transform $\mathcal{F} : L^2(\mathbf{T}^2) \rightarrow l^2(\mathbf{Z}^2)$ yields the operators

$$\begin{aligned} \hat{U}_\hbar f(x, y) &= e^{ix} f\left(x, y - \frac{\hbar}{2}\right) \\ \hat{V}_\hbar f(x, y) &= e^{iy} f\left(x + \frac{\hbar}{2}, y\right) \end{aligned}$$

on $L^2(\mathbf{T}^2)$, with $W^*(\hat{U}_\hbar, \hat{V}_\hbar)$ reducing to the algebra of bounded multiplication operators when $\hbar = 0$. However, for our purposes the $l^2(\mathbf{Z}^2)$ picture is more convenient.

The following commutation relations will be useful. For $f \in l^\infty(\mathbf{Z}^2)$ and $k, l \in \mathbf{Z}$ let $\tau_{k,l}f$ be the translated function $\tau_{k,l}f(m, n) = f(m - k, n - l)$. Then

$$\begin{aligned} U_\hbar V_\hbar &= e^{-i\hbar} V_\hbar U_\hbar \\ U_\hbar^k V_\hbar^l M_f &= M_{\tau_{k,l}f} U_\hbar^k V_\hbar^l. \end{aligned}$$

In particular, $U_\hbar^k V_\hbar^l M_{e^{-i(mx+ny)}} = e^{i(kx+ly)} M_{e^{-i(mx+ny)}} U_\hbar^k V_\hbar^l$.

Our main technical tool will be a kind of Fourier analysis. We introduce the relevant definitions.

Definition 2.34. Let $A \in \mathcal{B}(l^2(\mathbf{Z}^2))$.

(a) For $x, y \in \mathbf{T}$ define

$$\theta_{x,y}(A) = M_{e^{i(mx+ny)}} A M_{e^{-i(mx+ny)}}.$$

(b) For $k, l \in \mathbf{Z}$ define

$$A_{k,l} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-i(kx+ly)} \theta_{x,y}(A) dx dy.$$

We call $A_{k,l}$ the (k, l) *Fourier term* of A .

(c) For $k, l \in \mathbf{N}$ define

$$S_{k,l}(A) = \sum_{|k'| \leq k, |l'| \leq l} A_{k',l'}$$

and for $N \in \mathbf{N}$ define

$$\sigma_N(A) = \frac{1}{N^2} \sum_{0 \leq k, l \leq N-1} S_{k,l}(A).$$

In the $L^2(\mathbf{T}^2)$ picture the operator $M_{e^{i(mx+ny)}}$ on $l^2(\mathbf{Z}^2)$ becomes translation by $(-x, -y)$, so that $\theta_{x,y}$ is conjugation by a translation.

The integral used to define $A_{k,l}$ can be understood in a weak sense: for any vectors $v, w \in l^2(\mathbf{Z}^2)$ we take $\langle A_{k,l}w, v \rangle$ to be $\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{-i(kx+ly)} \langle \theta_{x,y}(A)w, v \rangle dx dy$. In particular, if $w = e_{m,n}$ and $v = e_{m',n'}$ then we have

$$\langle A_{k,l}e_{m,n}, e_{m',n'} \rangle = \begin{cases} \langle Ae_{m,n}, e_{m',n'} \rangle & \text{if } m' = m+k \text{ and } n' = n+l \\ 0 & \text{otherwise.} \end{cases} \quad (*)$$

The $A_{k,l}$ are something like Fourier coefficients, the $S_{k,l}(A)$ like partial sums of a Fourier series, and the $\sigma_N(A)$ like Cesàro means. The next few results are minor reworkings of material in [20].

Proposition 2.35. *Let $A \in \mathcal{B}(l^2(\mathbf{Z}^2))$. Then $\|\sigma_N(A)\| \leq \|A\|$ for all N and $\sigma_N(A) \rightarrow A$ weak operator.*

Proof. We have

$$\sigma_N(A) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} K_N(x) K_N(y) \theta_{x,y}(A) dx dy$$

where K_N is the Fejér kernel,

$$K_N(x) = \sum_{n=-N+1}^{N-1} \left(1 - \frac{|n|}{N}\right) e^{inx} = \frac{1}{N} \left(\frac{\sin(Nx/2)}{\sin(x/2)} \right)^2.$$

Since $\|K_N\|_1 = 2\pi$, this shows that $\|\sigma_N(A)\| \leq \|A\|$, so the sequence $\{\sigma_N(A)\}$ is bounded. So it will suffice to check weak operator convergence against the vectors $e_{m,n}$. But if $|m' - m|, |n' - n| \leq N$ then

$$\langle \sigma_N(A)e_{m,n}, e_{m',n'} \rangle = \left(1 - \frac{|m' - m|}{N}\right) \left(1 - \frac{|n' - n|}{N}\right) \langle Ae_{m,n}, e_{m',n'} \rangle,$$

and this converges to $\langle Ae_{m,n}, e_{m',n'} \rangle$ as $N \rightarrow \infty$, as desired. \square

Lemma 2.36. (a) *For any $k, l \in \mathbf{Z}$ the map $A \mapsto A_{k,l}$ is weak* continuous from $\mathcal{B}(l^2(\mathbf{Z}^2))$ to $\mathcal{B}(l^2(\mathbf{Z}^2))$.*

(b) *Let $\mathcal{M} \cong l^\infty(\mathbf{Z}^2)$ be the von Neumann algebra of bounded multiplication operators in $\mathcal{B}(l^2(\mathbf{Z}^2))$. Then for any $A \in \mathcal{B}(l^2(\mathbf{Z}^2))$ and any $k, l \in \mathbf{Z}$ we have $A_{k,l} \in \mathcal{M} \cdot U_h^k V_h^l$.*

Proof. (a) By the Krein-Smulian theorem we need only check that if $A_\lambda \rightarrow A$ boundedly weak* then their (k, l) Fourier terms converge, for every k and l . Then since the net is bounded it is enough to check convergence against basis vectors, and this follows immediately from the formula $(*)$ above.

(b) A simple change of variable in the formula for $A_{0,0}$ shows that it commutes with the operator $M_{e^{i(mx+ny)}}$ for all $x, y \in \mathbf{T}$. But these operators generate the maximal abelian von Neumann algebra \mathcal{M} , so we must have $A_{0,0} \in \mathcal{M}$. The result for arbitrary k and l now follows from the observation that the $(0, 0)$ Fourier term of $AV_h^{-l}U_h^{-k}$ is $A_{k,l}V_h^{-l}U_h^{-k}$. \square

Proposition 2.37. *Let $A \in \mathcal{B}(l^2(\mathbf{Z}^2))$. Then $A \in W^*(U_{\hbar}, V_{\hbar})$ if and only if $A_{k,l}$ is a scalar multiple of $U_{\hbar}^k V_{\hbar}^l$ for all $k, l \in \mathbf{Z}$.*

Proof. Suppose $A = \sum \alpha_{k,l} U_{\hbar}^k V_{\hbar}^l$ is a polynomial in U_{\hbar} and V_{\hbar} . Then the formula (*) given above shows that $A_{k,l}$ equals $\alpha_{k,l} U_{\hbar}^k V_{\hbar}^l$ for all k and l , so that the (k, l) Fourier term of A is a scalar multiple of $U_{\hbar}^k V_{\hbar}^l$. The forward implication now follows for all $A \in W^*(U_{\hbar}, V_{\hbar})$ since the map $A \mapsto A_{k,l}$ is weak* continuous and the polynomials in U_{\hbar} and V_{\hbar} are weak* dense in $W^*(U_{\hbar}, V_{\hbar})$. Conversely, if every Fourier term belongs to $W^*(U_{\hbar}, V_{\hbar})$ then A must also belong to $W^*(U_{\hbar}, V_{\hbar})$ by Proposition 2.35. \square

Corollary 2.38. *The commutant of $W^*(U_{\hbar}, V_{\hbar})$ is $W^*(U_{-\hbar}, V_{-\hbar})$.*

Proof. A straightforward calculation shows that U_{\hbar} and V_{\hbar} each commute with both of $U_{-\hbar}$ and $V_{-\hbar}$. From this it easily follows that $W^*(U_{\hbar}, V_{\hbar})$ is contained in the commutant of $W^*(U_{-\hbar}, V_{-\hbar})$. Conversely, suppose $A \in \mathcal{B}(l^2(\mathbf{Z}^2))$ commutes with $U_{-\hbar}$ and $V_{-\hbar}$. For any $k, l, m, n \in \mathbf{Z}$ we have

$$\begin{aligned} \langle V_{-\hbar}^{-n} U_{-\hbar}^{-m} A e_{m,n}, e_{k,l} \rangle &= \langle A e_{m,n}, U_{-\hbar}^m V_{-\hbar}^n e_{k,l} \rangle \\ &= e^{i\hbar(nk - ml - mn)/2} \langle A e_{m,n}, e_{m+k, n+l} \rangle \end{aligned}$$

and

$$\langle A V_{-\hbar}^{-n} U_{-\hbar}^{-m} e_{m,n}, e_{k,l} \rangle = e^{-i\hbar mn/2} \langle A e_{0,0}, e_{k,l} \rangle.$$

Thus, letting $\alpha = e^{i\hbar kl/2} \langle A e_{0,0}, e_{k,l} \rangle$, we have

$$\begin{aligned} \langle A_{k,l} e_{m,n}, e_{m+k, n+l} \rangle &= \langle A e_{m,n}, e_{m+k, n+l} \rangle \\ &= e^{i\hbar(ml - nk)/2} \langle A e_{0,0}, e_{k,l} \rangle \\ &= \langle \alpha U_{\hbar}^k V_{\hbar}^l e_{m,n}, e_{m+k, n+l} \rangle. \end{aligned}$$

And since

$$\langle A_{k,l} e_{m,n}, e_{m', n'} \rangle = 0 = \langle \alpha U_{\hbar}^k V_{\hbar}^l e_{m,n}, e_{m', n'} \rangle$$

if either $m' \neq m + k$ or $n' \neq n + l$, we conclude that $A_{k,l} = \alpha U_{\hbar}^k V_{\hbar}^l$. Thus $A \in W^*(U_{\hbar}, V_{\hbar})$ by Proposition 2.37. This completes the proof. \square

With this background material in place, we now proceed to analyze translation invariant quantum relations on quantum tori.

Definition 2.39. (a) A quantum relation $\mathcal{V} \subseteq \mathcal{B}(l^2(\mathbf{Z}^2))$ on the quantum torus von Neumann algebra $W^*(U_{\hbar}, V_{\hbar})$ is *translation invariant* if $\theta_{x,y}(\mathcal{V}) = \mathcal{V}$ for all $x, y \in \mathbf{T}$.

(b) A subspace \mathcal{E} of the von Neumann algebra $\mathcal{M} \cong l^\infty(\mathbf{Z}^2)$ of bounded multiplication operators on $l^2(\mathbf{Z}^2)$ is *translation invariant* if

$$M_f \in \mathcal{E} \quad \Rightarrow \quad M_{\tau_{k,l} f} \in \mathcal{E}$$

for all $k, l \in \mathbf{Z}$, where $\tau_{k,l} f$ is the translation operator defined just before Definition 2.34.

(The two notions of translation invariance are not directly related. One refers to invariance under an action of \mathbf{T}^2 , the other to invariance under an action of \mathbf{Z}^2 .)

First we indicate an equivalent formulation of translation invariance framed in terms of projections.

Proposition 2.40. *Let \mathcal{V} be a quantum relation on $W^*(U_{\hbar}, V_{\hbar})$ and let \mathcal{R} be the corresponding intrinsic quantum relation (Theorem 2.32). Then \mathcal{V} is translation invariant if and only if*

$$(P, Q) \in \mathcal{R} \iff ((\theta_{x,y} \otimes I)(P), (\theta_{x,y} \otimes I)(Q)) \in \mathcal{R}$$

for all $x, y \in \mathbf{T}$ and all projections $P, Q \in W^*(U_{\hbar}, V_{\hbar}) \overline{\otimes} \mathcal{B}(l^2)$.

Proof. The forward implication holds because

$$(\theta_{x,y} \otimes I)(P(A \otimes I)Q) = (\theta_{x,y} \otimes I)(P)(\theta_{x,y}(A) \otimes I)(\theta_{x,y} \otimes I)(Q),$$

so that translation invariance of \mathcal{V} implies that there exists $A \in \mathcal{V}$ such that $P(A \otimes I)Q \neq 0$ if and only if there exists $A \in \mathcal{V}$ such that $(\theta_{x,y} \otimes I)(P)(A \otimes I)(\theta_{x,y} \otimes I)(Q) \neq 0$. Conversely, suppose \mathcal{V} is not translation invariant and find $A \in \mathcal{V}$ and $x, y \in \mathbf{T}$ such that $\theta_{x,y}(A) \notin \mathcal{V}$. By Lemma 2.8 we can find projections $P, Q \in W^*(U_{\hbar}, V_{\hbar}) \overline{\otimes} \mathcal{B}(l^2)$ such that $P(\theta_{x,y}(A) \otimes I)Q \neq 0$ but $P(B \otimes I)Q = 0$ for all $B \in \mathcal{V}$. Then $(P, Q) \notin \mathcal{R}$ but $((\theta_{-x,-y} \otimes I)(P), (\theta_{-x,-y} \otimes I)(Q)) \in \mathcal{R}$ because

$$(\theta_{-x,-y} \otimes I)(P)(A \otimes I)(\theta_{-x,-y} \otimes I)(Q) = (\theta_{-x,-y} \otimes I)(P(\theta_{x,y}(A) \otimes I)Q) \neq 0.$$

This proves the reverse implication. \square

Theorem 2.41. *Let $\mathcal{M} \cong l^\infty(\mathbf{Z}^2)$ be the von Neumann algebra of bounded multiplication operators in $\mathcal{B}(l^2(\mathbf{Z}^2))$ and let \mathcal{E} be a weak* closed, translation invariant subspace of \mathcal{M} . Then*

$$\mathcal{V}_{\mathcal{E}} = \{A \in \mathcal{B}(l^2(\mathbf{Z}^2)) : A_{k,l} \in \mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l \text{ for all } k, l \in \mathbf{Z}\}$$

is a translation invariant quantum relation on $W^*(U_{\hbar}, V_{\hbar})$. Every translation invariant quantum relation on $W^*(U_{\hbar}, V_{\hbar})$ is of this form.

Proof. Since \mathcal{E} is weak* closed, so is $\mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l$. Together with weak* continuity of the map $A \mapsto A_{k,l}$ (Lemma 2.36 (a)), this implies that $\mathcal{V}_{\mathcal{E}}$ is weak* closed. $\mathcal{V}_{\mathcal{E}}$ is clearly a linear subspace of $\mathcal{B}(l^2(\mathbf{Z}^2))$. To see that it is a bimodule over $W^*(U_{-\hbar}, V_{-\hbar})$ it suffices by weak* continuity to demonstrate stability under left and right multiplication by monomials in $U_{-\hbar}$ and $V_{-\hbar}$; this holds because the (k, l) Fourier term of $A U_{-\hbar}^m V_{-\hbar}^n$ is $A_{k-m, l-n} U_{-\hbar}^m V_{-\hbar}^n$, and the (k, l) Fourier term of $U_{-\hbar}^m V_{-\hbar}^n A$ is $U_{-\hbar}^m V_{-\hbar}^n A_{k-m, l-n}$. So if $A \in \mathcal{V}_{\mathcal{E}}$ then $A_{k-m, l-n} \in \mathcal{E} \cdot U_{-\hbar}^{k-m} V_{-\hbar}^{l-n}$, say $A_{k-m, l-n} = M_f U_{-\hbar}^{k-m} V_{-\hbar}^{l-n}$, and the (k, l) Fourier term of $A U_{-\hbar}^m V_{-\hbar}^n$ is

$$M_f U_{-\hbar}^{k-m} V_{-\hbar}^{l-n} U_{-\hbar}^m V_{-\hbar}^n = e^{i\hbar(nm-lm)} M_f U_{-\hbar}^k V_{-\hbar}^l \in \mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l,$$

while the (k, l) Fourier term of $U_{-\hbar}^m V_{-\hbar}^n A$ is

$$U_{-\hbar}^m V_{-\hbar}^n M_f U_{-\hbar}^{k-m} V_{-\hbar}^{l-n} = e^{i\hbar(nm-nk)} M_{\tau_{m,n}f} U_{-\hbar}^k V_{-\hbar}^l \in \mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l$$

since \mathcal{E} is translation invariant. Finally, $\mathcal{V}_{\mathcal{E}}$ is translation invariant because the (k, l) Fourier term of $\theta_{x,y}(A)$ is $\theta_{x,y}(A_{k,l})$, so if $A \in \mathcal{V}_{\mathcal{E}}$ then $A_{k,l} \in \mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l$, say $A_{k,l} = M_f U_{-\hbar}^k V_{-\hbar}^l$, and the (k, l) Fourier term of $\theta_{x,y}(A)$ is

$$\theta_{x,y}(M_f U_{-\hbar}^k V_{-\hbar}^l) = e^{i(kx+ly)} M_f U_{-\hbar}^k V_{-\hbar}^l \in \mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l.$$

So $\mathcal{V}_{\mathcal{E}}$ is a translation invariant quantum relation on $W^*(U_{\hbar}, V_{\hbar})$.

Now let \mathcal{V} be any translation invariant quantum relation on $W^*(U_{\hbar}, V_{\hbar})$ and let $\mathcal{E} = \mathcal{V} \cap \mathcal{M}$. Then \mathcal{E} is clearly a weak* closed subspace of \mathcal{M} , and it is translation invariant because $M_f \in \mathcal{E}$ implies

$$M_{\tau_{k,l}f} = U_{-\hbar}^k V_{-\hbar}^l M_f V_{-\hbar}^{-l} U_{-\hbar}^{-k} \in \mathcal{E}.$$

We claim that $\mathcal{V} = \mathcal{V}_{\mathcal{E}}$. To see this, first let $A \in \mathcal{V}$; then for any $k, l \in \mathbf{Z}$ we have $A_{k,l} \in \mathcal{V}$ by translation invariance and weak* closure of \mathcal{V} , and $A_{k,l} \in \mathcal{M} \cdot U_{-\hbar}^k V_{-\hbar}^l$ by Lemma 2.36, so $A_{k,l} V_{-\hbar}^{-l} U_{-\hbar}^{-k} \in \mathcal{E}$. This shows that $A \in \mathcal{V}_{\mathcal{E}}$ and we conclude that $\mathcal{V} \subseteq \mathcal{V}_{\mathcal{E}}$. Conversely, if $A \in \mathcal{V}_{\mathcal{E}}$ then $A_{k,l} \in \mathcal{E} \cdot U_{-\hbar}^k V_{-\hbar}^l \subseteq \mathcal{V}$ for all $k, l \in \mathbf{Z}$, and this implies that $A \in \mathcal{V}$ by Proposition 2.35. So $\mathcal{V}_{\mathcal{E}} \subseteq \mathcal{V}$. \square

Retaining the notation of Theorem 2.41, for any closed subset $S \subseteq \mathbf{T}^2$ the smallest weak* closed translation invariant subspace of \mathcal{M} that contains $M_{e^{i(mx+ny)}}$ if and only if $(x, y) \in S$ is

$$\begin{aligned} \mathcal{E}_0(S) &= \{M_f : f \in l^\infty(\mathbf{Z}^2) \text{ and } \sum f \bar{g} = 0 \text{ for all } g \in l^1(\mathbf{Z}^2) \text{ such that } \hat{g}|_S = 0\} \\ &= \overline{\text{span}}^{wk*} \{M_{e^{i(mx+ny)}} : (x, y) \in S\} \end{aligned}$$

and the largest is

$$\begin{aligned} \mathcal{E}_1(S) &= \{M_f : f \in l^\infty(\mathbf{Z}^2) \text{ and } \sum f \bar{g} = 0 \text{ for all } g \in l^1(\mathbf{Z}^2) \text{ such that } \hat{g}|_T = 0 \\ &\quad \text{for some neighborhood } T \text{ of } S\} \\ &= \bigcap_{\epsilon > 0} \mathcal{E}_0(N_\epsilon(S)) \end{aligned}$$

where $N_\epsilon(S)$ is the open ϵ -neighborhood of S (see, e.g., Section 3.6.16 of [12]).

Now for any weak* closed translation invariant subspace \mathcal{E} of \mathcal{M} let

$$S(\mathcal{E}) = \{(x, y) \in \mathbf{T}^2 : M_{e^{i(mx+ny)}} \in \mathcal{E}\}.$$

Then $S(\mathcal{E})$ is a closed subset of \mathbf{T}^2 and $\mathcal{E}_0(S) \subseteq \mathcal{E} \subseteq \mathcal{E}_1(S)$, and we immediately infer the following corollary.

Corollary 2.42. *Let $\mathcal{M} \cong l^\infty(\mathbf{Z}^2)$ be the von Neumann algebra of bounded multiplication operators in $\mathcal{B}(l^2(\mathbf{Z}^2))$, let \mathcal{V} be a translation invariant quantum relation on $W^*(U_{\hbar}, V_{\hbar})$, and let $\mathcal{E} = \mathcal{V} \cap \mathcal{M}$. Then*

$$\mathcal{V}_{\mathcal{E}_0(S)} \subseteq \mathcal{V} \subseteq \mathcal{V}_{\mathcal{E}_1(S)}$$

where $S = S(\mathcal{E})$ and $\mathcal{V}_{\mathcal{E}}$ is as in Theorem 2.41.

In particular, if $S(\mathcal{E})$ is a set of spectral synthesis then $\mathcal{E}_0(S) = \mathcal{E}_1(S)$ and hence $\mathcal{V} = \mathcal{V}_{\mathcal{E}_0(S)} = \mathcal{V}_{\mathcal{E}_1(S)}$.

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